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HIGHER ALGEBRA

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PREFACE

This work is designed as a text-book in universities, colleges, and technical schools, the first fifteen chapters being also adapted to use in high schools and academies by students who have some knowledge of elementary algebra.

The demonstrations constitute one of the characteristic features of the book. While most of our text-books on Algebra state with great clearness the theorems and rules, few of them, especially in the earlier parts, give the demonstrations in a way that enables a student to reproduce them. Usually illustration, explanation, and general demonstration are so intermingled that the student is not able to gather up and give in logical form just what constitutes the proof. In this work the plan is that which gives so much definiteness to our teaching in Geometry: each general principle is followed by a concise, logical demonstration, containing only the reasoning necessary to establish it, while all illustrations and explanations by special cases are given in separate articles. The student thus soon learns to know what is demanded in a general proof, and to distinguish between rigorous demonstration and verification or illustration by a special case. Without any loss of conclusiveness in reasoning, the methods employed have permitted, in many cases, much shorter and more easily followed demonstrations than those usually given.

Another characteristic feature is the substitution of short processes for many of the long and tedious ones in common use. As mathematical operations, at best, involve much drudgery, all practical means of shortening the work should be made available to the student. The few short processes given in our text-books are reserved until the student has formed a habit of using the long processes, and, consequently, he never gains a practical use of even these few. In this book short processes are introduced

at the beginning of the respective subjects, and are used wherever applicable. For example, a very expeditious method of multiplying by a factor of the form $x \pm a$ or $x \pm ay$ is fully explained, and is applied to successive multiplication, to putting together the factors that constitute the highest common divisor and the lowest common multiple, to forming an equation having given roots, and to various other operations. Again, the equally expeditious method of dividing by a factor of the form $x \pm a$ or $x \pm ay$, instead of being confined to finding the commensurable roots of higher equations, is fully explained in Division, and is applied to successive division, to factoring polynomials, to finding the highest common divisor and the lowest common multiple, to reducing fractions to their lowest terms, to finding the commensurable roots of higher equations, and to various other operations. The student thus becomes very familiar with the processes, and expert in their use; and he not only escapes much tedious labor that has no disciplinary value, but secures greater accuracy in his results, the shorter processes diminishing the liability to errors.

The methods given of obtaining at once the square root and the cube root of polynomials, without writing any intermediate steps, will be found not only a great relief from the tedious processes given in our books, but an exhilarating exercise in which students will take great interest.

The subject of Maxima and Minima of Functions is presented in a fuller and more systematic way than heretofore, and the application made to practical problems cannot fail to interest the student.

While those properties of higher equations which serve no useful purpose have been excluded, the subject has received a fuller treatment than is common. The reader will find many features presented in new and simpler ways, with everything leading toward the easiest and most expeditious methods of finding the roots of numerical higher equations. The process of finding the roots of equations having only even or only odd powers will be seen to be remarkably brief.

Differentiation of algebraic and logarithmic functions is intro-

duced, because it enables us to give in their true relations and with their proper significance differential coefficients, which are usually disguised under the name of "derived polynomials"; because it enables us to include Taylor's Formula, and thus avoid the usual cumbersome and unsatisfactory method of demonstrating the binomial formula and the logarithmic series; because it puts into its true relation $f'(x)$ in the theorem for multiple roots and in Sturm's Theorem; and because in his subsequent course in mathematics the student will not use the methods which differentiation avoids, and will constantly use the methods which it introduces. Differentiation is given by the fluxionary system, as furnishing the clearest and most satisfactory conceptions and the simplest demonstrations.

Many exercises and problems have been given in all the different subjects. These will be found well graded, and neither so difficult as to discourage the student nor so easy as to afford little discipline. To guide the student in the application of principles to numerical examples, such suggestions, observations, and model solutions as long experience in teaching the subject has shown to be needful have been given. While many of the exercises and problems are new, free use has been made of the various collections available.

The subject of Determinants is not included, because inexpensive texts devoted exclusively to the subject can be readily procured by the comparatively small number of students who have occasion to enter upon the subject at this stage of their progress. The chapter on the Theory of Functions will be found of more value to the student than would be an elementary treatment of the Loci of Equations, as it gives him needed practice in the interpretation of analytical results, and thus prepares him for the subject of Loci when he reaches it in Analytical Geometry.

Assistance has been received from so many sources that no attempt is made to name them. While in many respects the book is a wide departure from the texts of the day, nothing has been made different for the sake of novelty. Whether new or old, the methods which long experience with large classes has proved to be the best have been given.

The book is put forth with the hope that it will not only serve well the usual purpose of a text on Algebra, but that it will, in addition, cultivate in students much greater facility in the explanation of principles, and that it will relieve them from a great burden of worse than needless details in operations, which many generations of students have been required to carry.

CONTENTS

HIGHER ALGEBRA

CHAPTER I

DEFINITIONS AND NOTATION

1. **Quantity** is amount or extent, and is expressed in terms of a unit of the same kind.

Thus, 10, 5 bushels, 6 tons, 50 miles, 7 years, 100 dollars, m square feet, n cubic yards, are quantities, the units in order being the abstract number 1, 1 bushel, 1 ton, 1 mile, 1 year, 1 dollar, 1 square foot, 1 cubic yard.

2. Two or more quantities of the same kind are **Commensurable** or **Incommensurable** with reference to one another according as they can or can not be measured with the same unit.

Thus, if the sides of a rectangle are 3 feet and 4 feet respectively, the diagonal is 5 feet, and is, therefore, commensurable with the sides; but the diagonal of a square is incommensurable with its sides, being $\sqrt{2}$ times one of the sides.

3. **An Incommensurable Quantity,** without comparison with another quantity, is one that cannot be exactly expressed in the decimal notation.

Thus, $\sqrt{5}$ is an incommensurable quantity, being 2 plus a decimal fraction which never terminates.

4. **Algebra** is that branch of pure mathematics which treats of numbers as expressed by symbols having general values, and of the nature, transformations, and use of equations.

5. Algebra differs from arithmetic in the following important particulars:

1st. In arithmetic values are counted in only one direction from 0; while in algebra, by means of positive and negative quantities, values are counted in two opposite directions from 0.

2d. Arithmetical quantities, denoted by figures, have each a single, definite value; while algebraical quantities, represented by letters, may have any value we choose to assign to them. These quantities can be recognized anywhere in the operation, and the results are, therefore, general formulæ instead of special answers.

3d. In arithmetic a problem is solved by analyzing it, step by step; while in algebra the conditions are expressed in equations involving one or more unknown quantities, usually representing the answer or answers.

6. Algebra employs five kinds of symbols, viz. of quantity, of operation, of relation, of aggregation, and of abbreviation.

7. The Symbols of Quantity commonly employed are the following:

1st. The Arabic figures.

2d. The letters of the Roman alphabet, known quantities being represented by the leading letters, and unknown quantities by the final letters.

Similar quantities employed in the demonstration of a theorem or the solution of a problem are often represented by the same letter with different accents, as a', a'', a''', etc., read "a prime," "a second," "a third," etc., or by the same letter with different subscripts, as a_1, a_2, a_3, etc., read "a sub-one," "a sub-two," "a sub-three," etc.

Initial letters are sometimes used, as r for radius, s for sum, d for difference, etc.

The Greek π is used for the ratio of the circumference of a circle to its diameter.

3d. Zero, 0, used to denote not only the absence of value, but also a quantity that is less than any assignable value.

4th. Infinity, ∞, used to represent a quantity that is greater than any assignable value.

8. **The Symbols of Operation** in algebra are those common to all branches of mathematics, the principal ones being the following:

1st. The sign of addition, $+$, read "plus."

2d. The sign of subtraction, $-$, read "minus."

3d. The sign of multiplication, $\times$, read "times," "into," or "multiplied by."

Multiplication is indicated also by a dot between the quantities.

The quantities between which multiplication is indicated are called *Factors*, and the result of the multiplication is called the *Product*.

4th. The sign of division, $\div$, read "divided by."

Division is indicated also by writing the dividend above, and the divisor below, a line, in the fractional form.

5th. The sign of evolution, $\sqrt{\ }$, called the radical sign, and read "the square root of," the square root, or second root, being one of the two equal factors into which a number is conceived to be resolved. The 3d, 4th, or nth root, by which is meant one of the 3, 4, or n equal factors into which a number is conceived to be resolved, is indicated by writing 3, 4, or n in the vertex of the angle of the sign, this number being called the *Index*.

9. Besides the above signs, or symbols of operation, there are certain positions of quantities with reference to other quantities that indicate operations, the principal ones being the following:

1st. **A Coefficient**, which is a quantity written beside another quantity to show how many times the latter is taken.

When algebraic quantities are written in succession with no sign between them, their product is signified, and any factor, or the product of any number of factors, is the coefficient of the product of the remaining factors.

Thus, in $5\,mnx$, 5 is the coefficient of mnx, $5\,m$ is the coefficient of nx, and $5\,mn$ is the coefficient of x.

The name is usually applied to the numerical factor, and when this is 1, it is not written.

2d. **An Exponent**, which is a quantity written at the right of and above another quantity, its meaning being as follows:

(*a*) *When an exponent is a positive integer*, it indicates that the quantity affected by it is to be taken as a factor as many times as there are units in the exponent. The result of the multiplication is called a *Power;* hence a positive integral exponent is said to indicate a power.

Thus, $a \cdot a = a^2$, read "a square," "a 2d power," or "a 2d";
$a \cdot a \cdot a = a^3$, read "a cube," "a 3d power," or "a 3d";
$a \cdot a \cdot a \cdot a = a^4$, read "$a$ 4th power," or "a 4th";
$a \cdot a \cdot a \cdots$ to n factors $= a^n$, read "a nth power," or "a nth."

(*b*) *When an exponent is a positive fraction*,* the numerator indicates a power and the denominator a root of the quantity affected by it.

Thus, $32^{\frac{4}{5}}$ is the 4th power of the 5th root of 32, or the 5th root of the 4th power of 32.

(*c*) *When an exponent is negative*, it indicates the reciprocal of what it would indicate if it were positive, the reciprocal of a quantity being 1 divided by that quantity.

Thus, $a^{-2} = \frac{1}{a^2}$ and $a^2b^{-3} = \frac{a^2}{b^3}$.

10. The Symbols of Relation are the following:

1st. The sign of equality, $=$, read "equals," or "is equal to."

2d. The signs of inequality, $>$ and $<$, read respectively, "is greater than," and "is less than."

The signs, $\neq$, $\ngtr$, and $\nless$, read respectively, "is not equal to," "is not greater than," and "is not less than," are also employed to some extent.

3d. The sign of geometrical ratio, :, read "to," or "is to."

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* It will be shown in Art. 147 that if we assume the index law as proved for positive integral exponents to be general, the meaning of positive fractional exponents and negative exponents *must* be as defined above. Until that article is reached, their meaning will be treated as a matter of definition.

4th. The sign of equality of ratios, ::, read "as," or "equals."

Thus, $a:b::c:d$ is read either "a is to b as c is to d," or "the ratio of a to b equals the ratio of c to d."

5th. The sign of variation, $\propto$, read "varies as."

11. The Symbols of Aggregation are the parentheses ( ), the brackets [ ], the braces { }, and the vinculum ——, and all indicate that the algebraic expression included is to be treated as a whole.

Thus, $a-(b+c)$, $a-[b+c]$, $a-\{b+c\}$, and $a-\overline{b+c}$ all indicate that the sum of b and c is to be subtracted from a.

The vinculum is sometimes vertical.

Thus, $\left.\begin{array}{c} a \\ b \\ c \end{array}\right| x$ is the same as $\overline{a+b+c} \times x$.

The line between the numerator and denominator of a fraction has the effect of a vinculum.

Thus, $a-\dfrac{b+c}{5}$ is the same as $a-\frac{1}{5}(b+c)$.

12. The Symbols of Abbreviation are the following:

1st. The signs of deduction, $\therefore$ and $\because$, read respectively, "therefore" or "hence," and "since" or "because."

2d. The sign of continuation, $\cdots$, read "and so on," or "and so on to."

Thus, $1+3+5+7+\cdots$ is read "1 plus 3 plus 5 plus 7 and so on," and $a+ar+ar^2+\cdots ar^{n-1}$ is read "a plus ar plus ar^2 and so on to ar^{n-1}."

3d. The factorial sign, $\underline{|\ \ }$, which indicates the product of all the integers from 1 to the number written in the angle inclusive.

Thus, $\underline{|3}$ means 2×3, $\underline{|5}$ means $2 \times 3 \times 4 \times 5$, $\underline{|n}$ means $2 \times 3 \times 4 \cdots n$.

13. An Algebraic Expression is a quantity, simple or made up of parts, expressed in algebraic symbols.

14. **The Terms** of an algebraic expression are the parts connected by the signs + and −.

15. An algebraic expression is called a *Monomial*, a *Binomial*, a *Trinomial*, a *Quadrinomial*, or a *Polynomial*, according as it consists of one, two, three, four, or many terms.

The term *Polynomial* is applied also, in a general way, to any algebraic expression consisting of more than one term.

16. **Similar Terms** are terms having the same literal quantities affected with the same exponents.

Similar terms can differ, therefore, only in their numerical coefficients.

17. **Positive and Negative Quantities.** The signs + and −, besides indicating the operations of addition and subtraction, are used to distinguish quantities of opposite character.

Thus, if distance east is considered +, or positive, distance west must be considered −, or negative. If gains are +, losses are −. If temperature above 0 is +, temperature below 0 is −. If the latitude of a place north of the equator is +, that of a place south of the equator is −, while that of a place on the equator is 0. If a force tending to move a body in one direction is +, a force tending to move it in the opposite direction is −. And, in general, *quantities which contribute to one result being considered positive, those which contribute to the opposite result must be considered negative.*

As quantities on one side of 0 are +, and those on the opposite side of 0 are −, positive quantities are sometimes said to be greater than 0, and negative quantities less than 0.

For example, if a man possesses nothing and owes one hundred dollars, he must earn one hundred dollars before his capital can be expressed by 0, and we say he has a hundred dollars less than nothing.

An increase in the numerical value of a positive quantity increases its algebraic value; but an increase in the numerical value of a negative quantity decreases its algebraic value.

18. **An Axiom** is a truth which is assumed as self-evident.

19. **A Theorem** is a formal statement of a truth requiring proof.

20. **The Hypothesis** of a theorem consists of the conditions on which it is affirmed.

21. **A Problem** is a question proposed for solution.

22. **The Solution** of a problem is the process of obtaining the result sought.

23. **A Rule** directs how to proceed in solving problems that belong to the same class.

24. **A Demonstration** is the course of reasoning by which the truth of a theorem, or the correctness of a solution or a rule, is established.

25. **A Corollary** is an inference from a preceding theorem, demonstration, or solution.

26. **A Scholium** is a remark upon some feature of what has preceded.

CHAPTER II

ADDITION

27. The Algebraic Sum of several quantities is the excess of the positive over the negative, or of the negative over the positive quantities.

Thus, if a man's assets are \$5000 and his liabilities \$3000, the excess of his assets over his liabilities is \$2000. Now, if we consider assets positive and liabilities negative, the algebraic sum of his assets, + \$5000, and his liabilities, − \$3000, is + \$2000.

Again, if a force of 50 pounds is exerted to move a body in one direction, and a force of 30 pounds to move it in the opposite direction, the effective force, *i.e.* the aggregate or the algebraic sum of the two forces, is 20 pounds.

In the same way the algebraic sum of $+8$, $+3$, -5, $+2$, and -4 is 4, and the algebraic sum of $+7$, -6, -9, and $+1$ is -7.

28. Algebraic Addition is the process of finding the algebraic sum of several quantities.

29. Prob. *To add monomials.*

Rule. *1st. If the quantities are similar (Art. 16), find the algebraic sum of the coefficients and annex the common literal part.*

2d. If the quantities are dissimilar, write them in succession with their signs unchanged.

Dem. 1st. Let it be required to add

$$5x^2y,\ 7x^2y,\ -4x^2y, \text{ and } -2x^2y.$$

By Art. 9, 1st, $5x^2y$ is 5 times x^2y, and $7x^2y$ is 7 times the same quantity. Now, 5 times any quantity and 7 times the same quantity are 12 times that quantity, giving, in this case, $12x^2y$. By Art. 17, $-4x^2y$ is so opposed in character to $+12x^2y$ as to destroy or neutralize 4 of the 12 times x^2y, giving $+8x^2y$. Similarly, $-2x^2y$ destroys 2 of the 8 times x^2y, giving for the sum of the four terms $+6x^2y$, in which the common literal part is

simply annexed to the algebraic sum of the coefficients. The same reasoning applies to any other set of quantities.

2d. Let it be required to add $+3a$, $+4b$, $-2c$, and $-d$.

Now, $3a$ and $4b$ are not respectively 3 and 4 times the same quantity; hence the sum will not be 7 times either quantity, and we can only indicate the addition, giving for the sum of the two $3a+4b$. For the same reason the addition of $-2c$ and $-d$ to this sum can only be indicated, the minus signs being preserved, since these negative terms neutralize a part of the positive sum. Hence the entire sum is

$$3a+4b-2c-d,$$

in which the given quantities are written in succession with their signs unchanged.

30. Cor. *Adding a negative quantity is the same as subtracting a numerically equal positive quantity.*

31. Prob. *To add polynomials.*

Rule. *Write the quantities so that similar terms shall be in the same vertical column. Add the columns separately and connect their sums by the resulting signs.*

Dem. The columns may be added separately by Art. 29, 1st. The sums being dissimilar, the addition of them, according to Art. 29, 2d, can only be indicated by connecting them by the resulting signs.

32. In writing several similar terms in column for convenience in adding, much time is saved by writing the literal factors only once. The plus signs also may be omitted, inasmuch as when no sign is written, the plus sign is understood.

$$\text{Instead of}\left\{\begin{array}{l} +7a^3b^2c^4\\ -3a^3b^2c^4\\ -5a^3b^2c^4\\ +2a^3b^2c^4\\ -6a^3b^2c^4\\ +9a^3b^2c^4\\ \hline +4a^3b^2c^4\end{array}\right. \text{write}\left\{\begin{array}{r|l} 7 & a^3b^2c^4\\ -3 & \\ -5 & \\ 2 & \\ -6 & \\ 9 & \\ \hline \multicolumn{2}{c}{4a^3b^2c^4}\end{array}\right. \text{or}\left\{\begin{array}{l} \;\;7a^3b^2c^4\\ -3\\ -5\\ \;\;2\\ -6\\ \;\;9\\ \hline \;\;4a^3b^2c^4\end{array}\right.$$

While the first operation, exclusive of the answer, contains 48 figures, letters, and signs, either of the others contains but 15.

33. Note. Although the principles of fractions will be treated in their proper place, we shall from the beginning assume such knowledge of them as comes from the study of Arithmetic.

EXAMPLES I

Add the following:

1. $2x^2 - 3x^2y - 4xy^2 + 6y^2$, $4x^2y - xy^2 - 5x^3$, $4x^2 - 5x^2y - y^3 + 2x^3 - 3y^2$, $x^2 + 8y^3 - 2y^2 + 7xy^2$, $2x^2y + 3xy^2 - 3y^3$, and $2x^3 - 4y^3 + 3y^2 + x^2$.

Operation

$$\begin{array}{llllll}
2x^2 & -\,3x^2y & -\,4xy^2 & +\,6y^2 & & \\
 & 4 & -1 & & -\,5x^3 & \\
4 & -5 & & -3 & 2 & -\,y^3 \\
1 & & 7 & -2 & & 8 \\
 & 2 & 3 & & & -3 \\
1 & & & 3 & 2 & -4 \\
\hline
8x^2 & -\,2x^2y & +\,5xy^2 & +\,4y^2 & -\,x^3 &
\end{array}$$

2. $m^2 - 3mn + 2n^2$, $3n^2 - m^2$, and $5mn - 3n^2 + 2m^2$.

3. $2ab - 3ax^2 + 2a^2x$, $12ab + 10ax^2 - 6a^2x$, and $ax^3 - 8ab - 5a^2x$.

4. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$, $4x^3y - 12x^2y^2 + 12xy^3 - 4y^4$, $6x^2y^2 - 12xy^3 + 6y^4$, $4xy^3 - 4y^4$, and y^4.

5. $5a^2cx^2 + 4a^2bx^2 + mx^2y^2$ and $10a^2cx^2 - 2a^2bx^2 + 6mx^2y^2$.

6. $3ab^2 - 4a^2b + a^3$, $5ab^2 - 4ac^2 - c^3$, and $2a^2b - 7ab^2 - 6ac$.

7. $2a^2 + 5ab - xy$, $-7a^2 + 3ab - 3xy$, $-3a^2 - 7ab + 5xy$, and $9a^2 - ab - 2xy$.

8. $5a^3b^2 - 8a^2b^3 + x^2y + xy^2$, $4a^2b^3 - 7a^3b^2 - 3xy^2 + 6x^2y$, $3a^3b^2 + 3a^2b^3 - 3x^2y + 5xy^2$, and $2a^2b^3 - a^3b^2 - 3x^2y - 3xy^2$.

9. $a^2 - b^2 + 3a^2b - 5ab^2$, $3a^2 - 4a^2b + 3b^3 - 3ab^2$, $a^3 + b^3 + 3a^2b$, $2a^3 - 4b^3 - 5ab^2$, $6a^2b + 10ab^2$, and $-6a^3 - 7a^2b + 4ab^2 + 2b^3$.

10. $2m^2 + \frac{1}{5}n^2 - \frac{2}{3}m^2n + 3mn^2 - mn$, $\frac{3}{5}n^2 - 2mn^2 + 3m^2n$, $m^2 - \frac{4}{5}n^2 + 4mn + \frac{1}{3}m^2n$, and $\frac{1}{2}mn^2 + 3m^2 - 3mn + 2n^2$.

11. $2x^{\frac{1}{2}} - 4x^{\frac{1}{3}} + x^2$, $5x^2y - ab + x^{\frac{1}{3}}$, $4x^2 - x^3$, and $2x^{\frac{1}{3}} - 3 + 2x^{\frac{1}{2}}$.

12. $ax + 2by + cz$, $\sqrt{x} + \sqrt{y} + \sqrt{z}$, $3\sqrt{y} - 2\sqrt{x} + 3\sqrt{z}$, $4cz - 3ax - 2by$, and $2ax - 4\sqrt{y} - 2\sqrt{z}$.

34. *Literal terms that are similar with reference to only part of their factors, may be united into one term with a polynomial coefficient.*

Thus, $7ax$, $-2bx$, and $3cx$ are similar with reference to x only, being, respectively, $7a$, $-2b$, and $3c$ times x, and the sum is $(7a - 2b + 3c)$ times x, or $(7a - 2b + 3c)x$.

35. *Compound terms that have a common compound, or polynomial factor may be added with reference to that factor.*

Thus, $5(x^2 - y)$, $3(x^2 - y)$, and $-2(x^2 - y)$ are similar with reference to the quantity $x^2 - y$, being, respectively, 5, 3, and -2 times $x^2 - y$, and the sum is $(5 + 3 - 2)$ times $(x^2 - y)$, or $6(x^2 - y)$.

EXAMPLES II

Add the following:

1. ax, bcx, $-3bx$, $-2cx$, and $4x$.

2. $ax - 2by$, $2bx - 3by$, and $cy - ax$.

3. $ax + 2by - 4z$, $3y - cz$, and $4x - 6y + 2cz$.

4. $a(x + y) + b(x - y)$ and $m(x + y) - n(x - y)$.

5. $a + b\sqrt{1 - c^2}$ and $a - b\sqrt{1 - c^2}$.

6. $5\sqrt{x - y}$, $5a\sqrt{x - y}$, $-3\sqrt{x - y}$, and $-a\sqrt{x - y}$.

7. $\frac{1}{2}\sqrt[3]{a^2 - x}$, $-\frac{2}{3}\sqrt[3]{a^2 - x}$, and $\sqrt[3]{a^2 - x}$.

8. $\dfrac{3a}{\sqrt{x}} + \dfrac{5}{y} - 2b$, $\dfrac{4a}{\sqrt{x}} - \dfrac{2}{y} + 8b$, and $-\dfrac{6a}{\sqrt{x}} - \dfrac{m}{y} - 3b$.

9. $(a + b - c)\sqrt{m^2 - x^2}$, $(a - b + c)\sqrt{m^2 - x^2}$, and $3a\sqrt{m^2 - x^2}$.

10. $(a + b - c)\sqrt[5]{x^2 - y^2}$, $(a - b + c)\sqrt[5]{x^2 - y^2}$, and $(b + c - a)\sqrt[5]{x^2 - y^2}$.

CHAPTER III

SUBTRACTION

36. **The Algebraic Difference** of two quantities is their difference with regard to their character as positive and negative, the order of subtraction being independent of their magnitudes.

While in finding the arithmetical difference the less quantity is always taken from the greater, both quantities being regarded as positive, in finding the algebraic difference either quantity may be taken from the other, and one or both of the quantities may be negative.

Thus, suppose that a merchant who had \$500 bought goods to the amount of \$700, and we are asked to state how much money he had left. In an arithmetical sense the question is absurd, inasmuch as he not only had nothing left, but went in debt \$200. In an algebraic sense, however, the question is not absurd, the answer being $-$ \$200.

Again, if one man has \$500 and owes nothing, and another man owes \$200 and has nothing, the difference of their possessions, a debt being regarded as a negative possession, is \$700.

37. **Algebraic Subtraction** is the process of finding the algebraic difference between two quantities.

38. It is seen that subtraction is the inverse of addition, the problem being to find one of two quantities when the other and the sum of the two are given.

39. The terms *Minuend* and *Subtrahend* are used in Algebra as in Arithmetic, but Addition, Subtraction, Sum, and Difference (or Remainder) have a more general signification. In Arithmetic addition always produces an increase, and subtraction a decrease; but in Algebra addition may produce a decrease and subtraction an increase.

40. Prob. *To subtract one quantity from another.*

RULE. *Conceive the signs of all the terms of the subtrahend to be changed, then proceed as in addition.*

DEM. Let it be required to subtract $m - n$ from a.

If the whole of m be subtracted from a, the result will be $a - m$. Now since the quantity subtracted is too large by n, the remainder is too small by n; hence to make it what it should be, we add n to the result, giving $a - m + n$. In this operation we see that we have changed the signs of the subtrahend and added the result to the minuend.

41. Cor. *Subtracting a negative quantity is the same as adding a numerically equal positive quantity.*

EXAMPLES III

From $3a^3 - 4a^2b + 5ab^2 - 6b^3$ take $a^3 + 2a^2b - 3ab^2 - 8b^3$.

OPERATION

$$\begin{array}{llll} 3a^3 & -4a^2b & +5ab^2 & -6 \\ 1 & \quad 2 & -3 & -8 \\ \hline 2a^3 & -6a^2b & +8ab^2 & +2b^3 \end{array}$$

2. From $3x^3 - 2x^2 - x - 7$ take $2x^3 - 3x^2 + x + 1$.

3. From $x^2 + 2xy + y^2$ take $x^2 - 2xy + y^2$.

4. From $1 + 3x + 3x^2 + x^3$ take $1 - 3x + 3x^2 - x^3$.

5. From $10x^3 - 8x^2 + 6x + 4$ take $7x^3 + 5x^2 - 3x + 7$.

6. From $3a - 3a^2 + 1 + a^3$ take $1 - a^3 - a - a^2$.

7. From $a^2 + 2ab + b^2$ take the sum of $-a^2 + 2ab - b^2$ and $-2a^2 + 2b^2$.

8. From $2x^4 + 4x^3y - 5x^2y^2 - 3xy^3 + y^4$ take $x^4 - x^3y - 3x^2y^2 + 6xy^3 + y^4$.

9. From the sum of $2a - 3b + 4d$ and $2b + 4c - 3d$ take the sum of $3c - 4a - 4b - 2d$ and $3a - 2c$.

10. From $1 + 3\sqrt{x} + 3x + \sqrt{x^3}$ take $1 - 3\sqrt{x} + 3x - \sqrt{x^3}$.

11. From $x^{\frac{4}{5}} + 2x^{\frac{2}{5}}y^{\frac{2}{5}} + y^{\frac{4}{5}}$ take $x^{\frac{4}{5}} - 2x^{\frac{2}{5}}y^{\frac{2}{5}} + y^{\frac{4}{5}}$.

12. From $5(x+y)+3(x-y)$ take $2(x+y)-3(x-y)$.

13. From $a+b\sqrt{1-c^2}$ take $a-b\sqrt{1-c^2}$.

14. Subtract $\frac{1}{8}\sqrt[3]{a+x^2}$ from $\frac{3}{4}\sqrt[3]{a+x^2}$.

15. Subtract $(b-a-c)\sqrt{x^2+y^2}$ from $(a-b+c)\sqrt{x^2+y^2}$.

16. Subtract $b\sqrt{x+y}-a\sqrt{x-y}$ from $a\sqrt{x+y}+b\sqrt{x-y}$.

17. What must be added to $4x^3-3x^2+2$ to produce $4x^3+7x-6$?

18. To what must $x^3+3x^2y+3xy^2+y^3$ be added to produce $3x^3-x^2y-2xy^2+4y^3$?

19. What must be subtracted from $5a-4b+3c$ to leave $2a+2b-c$?

20. From what must $\sqrt[3]{x+y}-b\sqrt{x+y}-d(x-y)$ be subtracted to leave $5\sqrt[3]{x+y}+a\sqrt{x+y}+c(x-y)$?

SIGNS OF AGGREGATION AS RELATED TO ADDITION AND SUBTRACTION

42. Theorem. *1st. A parenthesis, or other sign of aggregation, may be removed without changing the signs of the inclosed terms, if it is preceded by +, and by changing the signs of the inclosed terms, if it is preceded by −.*

2d. Conversely, terms may be inclosed within a parenthesis, or other sign of aggregation, without changing their signs, if it is preceded by +, and by changing their signs, if it is preceded by −.

DEM. 1st. A sign of aggregation indicates that the inclosed terms are to be treated as a whole (Art. 11); hence, if preceded by +, all the inclosed terms are to be added, which does not change their signs (Art. 29), and if preceded by −, all the terms are to be subtracted, which changes their signs (Art. 40).

2d. The truth of the converse is seen from the fact that the removal of the sign of aggregation by the preceding principle would restore the expression to its original form.

43. Sch. When signs of aggregation occur within other signs of aggregation, they may be removed in succession, and it is usually expedient to begin with the innermost.

Thus,
$$\begin{aligned} &a-[b+\{c-(d-\overline{e-f})\}] \\ &=a-[b+\{c-(d-e+f)\}] \\ &=a-[b+\{c-d+e-f\}] \\ &=a-[b+c-d+e-f] \\ &=a-b-c+d-e+f. \end{aligned}$$

EXAMPLES IV

Perform the following indicated operations:

1. $2x^2-\{5x-(x^2+2x-5)\}$.
2. $4a-3b-\{2b+3-(2a+b+1)\}$.
3. $7a-[3a-\{4a-(5a-2)\}]$.
4. $x^2-x-(2x-(3x^2+4-(x-1)))$.
5. $2m-[3m-\{m-(2m-\overline{3m+4})\}-(5m-2)]$.
6. $4z-(12x-2y)-\{2z-(10x+4y)-(2x-6y)\}$.
7. $x-[y+z-\{x-(-x-y)+z\}]+\{z-(2x-y)\}$.

In the following unite terms with reference to similar factors, preceding each polynomial coefficient by $-$:

8. $ax^3+2bx^2-x-3x^3+4x^2+5x-cx^2-2dx$.
9. $ax^2+a^2x^3-bx^2-5x^2-cx^3$.
10. $ax^3-2cx-[bx^2-\{cx-dx-(bx^3+3cx^2)\}-(cx^2-bx)]$.

CHAPTER IV

MULTIPLICATION

44. Multiplication is the process of finding a quantity which shall be as many times a specified quantity, or such a part of that quantity, as is represented by a specified number.

The terms *Multiplicand, Multiplier, Product,* and *Factors* are used as in Arithmetic.

It follows from the definition that multiplication by a positive number is the repeated addition of the whole or a part of the multiplicand to 0; while multiplication by a negative number is the repeated subtraction of the whole or a part of the multiplicand from 0.

Thus, $a \times b$ is a repeated b times, and $a \times \frac{m}{n}$ is one nth part of a repeated m times; while $a \times (-b)$ is a subtracted b times, and $a \times \left(-\frac{m}{n}\right)$ is one nth part of a subtracted m times.

The only difference between $\frac{a}{n} \times m$ and $a \times \frac{m}{n}$ is that in the former one nth part of a is represented as already taken, and this nth part is to be repeated m times; while in the latter one nth part of a is first to be taken, and then this nth part is to be repeated m times.

45. Theorem. *The product has the same sign as the multiplicand when the multiplier is positive, and the opposite sign when the multiplier is negative.*

Dem. Repeated addition of the whole or a part of the multiplicand would not change the sign (Art. 29), while repeated subtraction of the whole or a part of the multiplicand would change the sign (Art. 40).

46. Cor. I. *When there are but two factors, like signs give $+$, and unlike signs give $-$.*

47. Cor. 2. *The product of an even number of negative factors is +, and of an odd number −.*

48. Theorem. *The exponent of a quantity in the product is the sum of the exponents of the same quantity in the multiplicand and multiplier.*

Dem. 1st. When the exponents are positive integers.

Let it be required to multiply a^m by a^n. By definition (Art. 9, 2d)

$$\begin{aligned} a^m \times a^n &= aaaa \cdots \text{ to } m \text{ factors} \times aaaa \cdots \text{ to } n \text{ factors} \\ &= aaaaaa \cdots \text{ to } m+n \text{ factors} = a^{m+n}. \end{aligned}$$

2d. When the exponents are positive fractions.

Let it be required to multiply $a^{\frac{m}{n}}$ by $a^{\frac{p}{q}}$.

Now $a^{\frac{m}{n}} = a^{\frac{mq}{nq}}$; for while the denominator of the exponent in the latter indicates that a is resolved into q times as many factors as in the former, the numerator indicates that q times as many of these factors are taken. In like manner $a^{\frac{p}{q}} = a^{\frac{np}{nq}}$. Therefore,

$$a^{\frac{m}{n}} \times a^{\frac{p}{q}} = a^{\frac{mq}{nq}} \times a^{\frac{np}{nq}} = (a^{\frac{1}{nq}})^{mq} \times (a^{\frac{1}{nq}})^{np},$$

which, by the first part of the demonstration, is

$$(a^{\frac{1}{nq}})^{mq+np} = a^{\frac{mq+np}{nq}} = a^{\frac{m}{n}+\frac{p}{q}}.$$

3d. When the exponents are negative.

Let it be required to multiply a^{-m} by a^{-n}.
By definition (Art. 9, 2d)

$$a^{-m} \times a^{-n} = \frac{1}{a^m} \times \frac{1}{a^n}.$$

Now, as fractions are multiplied by multiplying the numerators together for a new numerator, and the denominators together for a new denominator, we have by the first part of the demonstration,

$$a^{-m} \times a^{-n} = \frac{1}{a^m} \times \frac{1}{a^n} = \frac{1}{a^{m+n}} = a^{-m-n}.$$

EXAMPLES V

Show that the following results are true, whether the multiplication is before or after performing the operations indicated by the exponents:

1. $81^{\frac{3}{2}} \times 81^{\frac{3}{4}} = 19683.$ **2.** $16^{-\frac{3}{4}} \times 16^{-\frac{1}{2}} = \frac{1}{32}.$

3. $25^{-\frac{1}{2}} \times 25^{\frac{1}{2}} = 1.$ **4.** $a^{-2} \times a^{3} = a.$

Find the products of the following:

5. $4\,a^2b^3$ and $5\,a^3b^2.$ **6.** $12\,a^{-3}bc^2$ and $6\,a^4b^{-2}c.$

7. $14\,x^4y^{\frac{1}{2}}$ and $-3\,x^{-1}y^{\frac{3}{4}}.$ **8.** $-20\,x^{\frac{2}{3}}y^{\frac{1}{2}}$ and $-8\,x^{\frac{1}{2}}y^{-\frac{2}{3}}.$

9. $5\,ab^2,\ -2\,a^2b^3,\ -bc,$ and $3\,a^2bc^2.$

10. $3\,ax,\ -3\,a^2x^2,\ 4\,by,\ -y^3,$ and $2\,x^2y^2.$

11. $7\,x^{\frac{1}{3}},\ x^{\frac{1}{2}}y,\ -2\,xy^{\frac{2}{3}},$ and $3\,x^{\frac{2}{3}}y^{\frac{1}{3}}.$

12. $3\,a^2b^{-3}c^{\frac{1}{2}},\ -4\,abc^{\frac{1}{3}},\ 2\,a^{-3}bc^{-1},$ and $9\,a^3b^2.$

13. $6\,x^my^n,\ -5\,x^2y,\ -x^ny^m,\ 3\,x^{-1}y^3,$ and $-2\,x^{2m}y^{3n}.$

49. Prob. *To multiply one polynomial by another.*

RULE. *Multiply each term of the multiplicand by each term of the multiplier and add the products.*

DEM. Let it be required to multiply any polynomial by $a + b - c$.

If we multiply it by a and then by b and add the results, we shall have taken it $a + b$ times. But the multiplier required that it be taken only $a + b$ *minus* c times. Hence from $a + b$ times the given polynomial we must deduct c times the same polynomial. But subtracting this product is the same as adding it with its signs changed, and these are changed by regarding the minus sign of c in multiplying.

50. The Degree of a Term is the sum of the exponents of its literal factors.

51. The Degree of a Polynomial is the same as that of its term of highest degree.

52. A polynomial is said to be **Arranged** with reference to a certain letter when the exponents of that letter increase or decrease in the successive terms.

53. A Homogeneous Polynomial is one in which all the terms are of the same degree.

54. Theorem. *The product of two homogeneous polynomials is itself homogeneous.*

Dem. If the degree of every term of the first polynomial is m and of the second n, then the product of any term of the first by any term of the second will be of the $(m+n)th$ degree. Hence the product will be homogeneous.

55. If multiplicand and multiplier are arranged with reference to the same letter, the product will have a similar arrangement, and the work will be more systematic than it otherwise would be. When terms are missing from an arranged multiplicand, their places should be supplied by terms with coefficient zero. For convenience in adding, similar terms of the partial products are written in the same column. *The literal factors should be written but once,* as the writing of a coefficient under another sufficiently indicates that the literal factors are the same.

EXAMPLES VI

Multiply the following:

1. $6x^3 - 4x^2y - 2xy^2 + 3y^3$ by $2x^2 - 3xy + 4y^2$.

Operation

$$
\begin{array}{llllll}
6x^3 & -\ 4x^2y & -\ 2xy^2 & +\ 3y^3 & & \\
2x^2 & -\ 3xy & +\ 4y^2 & & & \\
\hline
12x^5 & -\ 8x^4y & -\ 4x^3y^2 & +\ 6x^2y^3 & & \\
 & -\ 18 & \quad 12 & \quad 6 & -\ 9xy^4 & \\
 & & \quad 24 & -\ 16 & -\ 8 & \quad 12y^5 \\
\hline
12x^5 & -\ 26x^4y & +\ 32x^3y^2 & -\ 4x^2y^3 & -\ 17xy^4 & +\ 12y^5
\end{array}
$$

2. $3a^2 - 4ab + 6b^2$ by $2a^2 - 5ab + 3b^2$.

3. $3x^2 + 2x - 4$ by $x^2 - 3x + 2$.

4. $2x^3 + 4 - 3x - 3x^2$ by $3x - 2 + 4x^2$.

5. $x^4 - x^2y^2 + y^4$ by $x^2 + y^2$.

6. $x^2 + xy + y^2$ by $x^2 - xy + y^2$.

7. $m^4 + n^4 + p^4 - m^2n^2 - m^2p^2 - n^2p^2$ by $m^2 + n^2 + p^2$.

8. $a^m - a^n + a^2$ by $a^m - a$.

9. $4a^2 + 9b^2 + c^2 + 3bc + 2ac - 6ab$ by $2a + 3b - c$.

10. $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$.

11. $x^4 - 2x^2 + 3x - 5$ by $x^3 - 2x^2 + 3x - 4$.

Operation

$$
\begin{array}{lllllll}
x^4 + 0x^3 & -\,2x^2 & +\,3x & -\,5 & & & \\
x^3 - 2x^2 & +\,3x & -\,4 & & & & \\
\hline
x^7 + 0x^6 & -\,2x^5 & +\,3x^4 & -\,5x^3 & & & \\
\quad -2 & & 4 & -\,6 & 10x^2 & & \\
 & 3 & & -\,6 & 9 & -\,15x & \\
 & & -\,4 & & 8 & -\,12 & 20 \\
\hline
x^7 - 2x^6 & +\,x^5 & +\,3x^4 & -\,17x^3 & +\,27x^2 & -\,27x & +\,20
\end{array}
$$

12. $2x^4 + 3x^2 - 4$ by $x^3 + 5x^2 - 6x + 3$.

13. $3x^4 - 4x^2y^2 - 2xy^3 + 5y^4$ by $2x^2 - xy + 3y^2$.

14. $m^5 - 7m^3 + 4m - 8$ by $m^3 - 3m + 5$.

15. $x^4 + 3x^3 - 2x + 5$ by $x^3 - 2x^2 + 7x - 3$.

16. $x^5 + 4x^3 - 3$ by $3x^4 - 2x^2 + 4$.

17. $4x^{3p} - 3x^{2p}y^q + 6x^py^{2q} + 2y^{3q}$ by $2x^p + 3y^q$.

MULTIPLICATION BY DETACHED COEFFICIENTS

56. In examples like most of the foregoing, in which the terms of both multiplicand and multiplier contain the same letters, if the arrangement be made the same in both, the multiplication can be effected by using the coefficients alone, since the literal factors in the product will follow the same law of arrangement.

EXAMPLES VII

Multiply the following:

1. $6x^3 - 4x^2y - 2xy^2 + 3y^3$ by $2x^2 - 3xy + 4y^2$.

OPERATION

$$\begin{array}{rrrrrr}
6- & 4- & 2+ & 3 & & \\
2- & 3+ & 4 & & & \\
\hline
12- & 8- & 4 & 6 & & \\
 & -18 & 12 & 6- & 9 & \\
 & & 24 & -16- & 8 & 12 \\
\hline
12 & -26 & 32- & 4- & 17 & 12
\end{array}$$

Prod., $12x^5 - 26x^4y + 32x^3y^2 - 4x^2y^3 - 17xy^4 + 12y^5$.

2. $x^3 + 2x - 4$ by $x^2 - 1$.

OPERATION

$$\begin{array}{rrrrrr}
1+ & 0+ & 2- & 4 & & \\
1+ & 0- & 1 & & & \\
\hline
1 & 0 & 2- & 4 & & \\
 & & -1 & 0- & 2 & 4 \\
\hline
1 & 0 & 1- & 4- & 2 & 4
\end{array}$$

Prod., $x^5 + x^3 - 4x^2 - 2x + 4$.

3. $3x^2 - x + 2$ by $3x^2 + 2x - 2$.

4. $x^3 - 3x^2 + 3x - 1$ by $x^2 - 2x + 1$.

5. $3a^2 + 4ax - 5x^2$ by $2a^2 - 6ax + 4x^2$.

6. $27x^3 + 9x^2y + 3xy^2 + y^3$ by $3x - y$.

7. $2x^3 + 6x^2 - 4x - 3$ by $3x^3 - 4x^2 - x + 5$.

8. $m^3 + 2m^2n + 2mn^2$ by $m^2 - 2mn + 2n^2$.

9. $x^2 + 9x + 20$, $x^2 - 7x + 12$, and $x^2 - 2x - 15$.

10. $x^4 - x^2 + 1$, $x^2 + x + 1$, and $x^2 - x + 1$.

When the student writes down the polynomials to be multiplied, he should write them in the proper position for multiplying, and

then not make the unnecessary repetition of detaching the coefficients, but proceed at once as follows:

$$
\begin{array}{rrrrrr}
6x^3 - & 4x^2y - & 2xy^2 + & 3y^3 & & \\
x^2 - & 3xy + & 4y^2 & & & \\
\hline
12 & -8 & -4 & 6 & & \\
 & -18 & 12 & 6 & -9 & \\
 & & 24 & -16 & -8 & 12 \\
\hline
12x^5 - & 26x^4y + & 32x^3y^2 - & 4x^2y^3 - & 17xy^4 + & 12y^5
\end{array}
$$

11. $x^3 - 2ax^2 + 2a^2x - 3a^3$ by $x^2 - 3ax + 2a^2$.

12. $x^3 + 6x^2y + 12xy^2 + 8y^3$ by $x^3 - 3x^2y + 3xy^2 - y^3$.

13. $x^4 - 3x^3 + x^2 + 4x - 6$ by $2x^3 - 5x^2 - 3x + 5$.

14. $x^5 - 5x^4 + 13x^3 - x^2 - x + 2$ by $x^2 - 2x - 2$.

SHORT METHODS OF MULTIPLICATION

57. In many special cases the process of multiplication may be made very short, and in others the products may be written by inspection. These methods are of so much importance and save so much labor that they are here given by special rules and theorems. The student should make himself thoroughly familiar with them.

58. Prob. *To multiply by a binomial of the first degree, the coefficient of whose first term is unity, when multiplicand and multiplier contain but one letter, or when they contain two letters and are homogeneous.*

Rule. *Arrange the multiplicand in descending powers of the first letter of the multiplier, supplying the places of any missing terms by terms with coefficient zero. Call the numerical part of the second term a; multiply the coefficient of the first term of the multiplicand by a, and add the product to that of the second; multiply the coefficient of the second term of the multiplicand by a, and add the product to that of the third, and so on. The coefficient of the first term of the multiplicand and these sums in order will be the coefficients of the product, and the arrangement of the letters will be the same as in the multiplicand, the degree being one greater.*

DEM. Let it be required to multiply

$$3x^4 - 2x^3y - 5x^2y^2 + 4xy^3 - 6y^4 \text{ by } x + 3y.$$

By the process of Ex. 1, page 33, we have

$$\begin{array}{llllll}
3x^4 - 2x^3y - & 5x^2y^2 + & 4xy^3 - & 6y^4 & & \\
\;x + 3y & & & & & \\
\hline
3x^5 - 2x^4y - & 5x^3y^2 + & 4x^2y^3 - & 6xy^4 & & \\
\quad\;\; 9 & -6 & -15 & 12 & -18y^5 & \\
\hline
3x^5 + 7x^4y - & 11x^3y^2 - & 11x^2y^3 + & 6xy^4 - & 18y^5 &
\end{array}$$

Since the coefficient of the first term of the multiplier is, by hypothesis, unity, the coefficients of the terms of the first partial product will always be the same as those of the multiplicand.

The coefficients of the terms of the second partial product are a (in this case 3) times the coefficients of the terms of the multiplicand, and, being placed one term farther to the right to bring similar terms together, we see that to obtain the coefficients of the terms of the product, a (3) times the first coefficient of the multiplicand is added to the second, a (3) times the second is added to the third, and so on. Omitting the repetition of the coefficients of the terms of the multiplicand in the first partial product, the work stands as follows:

$$\begin{array}{lcccccc}
(x + 3y)(3x^4 - 2x^3y - & 5x^2y^2 + & 4xy^3 - & 6y^4) & & & \\
& 9 & -6 & -15 & 12 & -18 & \\
\hline
3 & 7 & -11 & -11 & 6 & -18 &
\end{array}$$

As shown before, the literal factors in the terms of the product follow the same law of arrangement as in the multiplicand, and since the multiplier is of the first degree, the degree of the product is one greater than that of the multiplicand. Hence supplying the literal parts we have for the product,

$$3x^5 + 7x^4y - 11x^3y^2 - 11x^2y^3 + 6xy^4 - 18y^5.$$

Only the a of the multiplier need be written; and when the coefficients are small, as in this case, or so related that the sums are

small, the additions can be readily made without writing the intermediate products.

$$
\begin{array}{llllll}
(3) & 3x^4 - 2x^3y - & 5x^2y^2 + & 4xy^3 - 6y^4 & & \\
 & & 7 & -11 & -11 & 6 \quad -18
\end{array}
$$

Thus, 3 times 3 are 9, which added to -2 gives 7; 3 times -2 are -6, which added to -5 gives -11; 3 times -5 are -15, which added to 4 gives -11; 3 times 4 are 12, which added to -6 gives 6; 3 times -6 are -18, which added to 0 gives -18.

Supplying the literal parts, we have for the product,

$$3x^5 + 7x^4y - 11x^3y^2 - 11x^2y^3 + 6xy^4 - 18y^5.$$

EXAMPLES VIII

Multiply the following:

1. $4x^5 - 3x^4y - 8x^3y^2 + 2x^2y^3 - 6xy^4 - 15y^5$ by $x - 6y$.

Operation

$$
\begin{array}{llllllll}
(-6) & 4x^5 - & 3x^4y - & 8x^3y^2 + & 2x^2y^3 - & 6xy^4 & -15y^5 & \\
 & & -27 & 10 & 50 & -18 & 21 & 90 \\
\textit{Prod.,} & \multicolumn{7}{l}{4x^6 - 27x^5y + 10x^4y^2 + 50x^3y^3 - 18x^2y^4 + 21xy^5 + 90y^6}
\end{array}
$$

If the student perform the multiplication in the usual way, writing the partial products in full, he will find that, between factors and product, he uses 70 figures, letters, and signs, while by this process he uses but 14.

2. $x^3 + 4x^2y + 7xy^2 - 5y^3$ by $x - 5y$.

3. $2x^4 - x^3y + 7x^2y^2 - 20xy^3 + 75y^4$ by $x + 4y$.

4. $4m^5 - 8m^4n + 16m^3n^2 - 16m^2n^3 + 8mn^4 - 4n^5$ by $m + 5n$.

5. $3m^4 - 7m^2n^2 + 6n^4$ by $m + 6n$.

Operation

$$
\begin{array}{llllll}
(6) & 3m^4 + & 0m^3n - & 7m^2n^2 + & 0mn^3 & +6n^4 \\
 & & 18 & -7 & -42 & 6 \quad 36 \\
\textit{Prod.,} & \multicolumn{5}{l}{3m^5 + 18m^4n - 7m^3n^2 - 42m^2n^3 + 6mn^4 + 36n^5}
\end{array}
$$

6. $5x^6 - 6x^4y^2 - 4x^2y^4 + 7y^6$ by $x - 3y$.

7. $x^4 - 4x^3y + 3xy^3 - 9y^4$ by $x + 7y$.

8. $a^4 - 4a^3b + 5a^2b^2 + 7ab^3 - 12b^4$ by $a + 12b$.

9. $3x^5 - 4x^4 + 6x^2 - 5x + 10$ by $x + 8$.

OPERATION

$$\begin{array}{llllllll} & (8) & 3x^5 - & 4x^4 + & 0x^3 + 6x^2 - & 5x & +10 & \\ & & & 20 & -32 & 6 & 43 \quad -30 & 80 \\ \textit{Prod.}, & & \multicolumn{6}{l}{3x^6 + 20x^5 - 32x^4 + 6x^3 + 43x^2 - 30x + 80} \end{array}$$

10. $2x^4 + 3x^3 + 4x^2 + 5x + 6$ by $x + 5$.

11. $4x^4 - 3x^3 - 2x^2 - x - 1$ by $x - 4$.

12. $x^5 - 3x^4 + 5x^3 + 2x^2 - 6x - 8$ by $x + 7$.

13. $x^5 - 8x^2 + 5$ by $x + 6$.

14. $3x^4 + 7$ by $x - 9$.

15. $5x^5 - 5$ by $x + 12$.

16. $x^6 + 3x^3 + 6$ by $x - 10$.

59. Successive Multiplication. When several factors like those of Art. 58 are to be combined into one product, the literal quantities should not be written until after all the multiplications have been made.

EXAMPLES IX

Find the products of the following:

1. $3x^2 - 2x + 1$, $x + 2$, $x - 1$, and $x + 3$.

OPERATION

$$\begin{array}{lllllll} (2) & 3x^2 - & 2x & +1 & & & \\ (-1) & & 4 & -3 & 2 & & \\ (3) & & 1 & -7 & 5 & -2 & \\ & & 10 & -4 & -16 & 13 & -6 \end{array}$$

Prod., $3x^5 + 10x^4 - 4x^3 - 16x^2 + 13x - 6$

2. $x^3 - 2x^2 + 4x - 8$, $x + 3$, and $x - 4$.

3. $x^2 - 3x + 5$, $x + 1$, and $x - 7$.

4. $x + 4$, $x - 3$, $x + 5$, and $x - 2$.

5. $2x^3-3x^2+4x-7,\ x+2,\ x-3,$ and $x+4.$

6. $x-7,\ x+1,\ x-2,\ x+3,\ x-4.$

7. $3x^2-4x+2,\ x+3,\ x-5,\ x-1,\ x+2.$

8. $3x+2,\ 3x-2,\ x+4,\ x-5,\ x-2,$ and $x+1.$

9. $x^2+2x+1,\ x+3,\ x+2,\ x-2,\ x-2,$ and $x-1.$

60. Theorem. *The product of two binomials having their first terms the same is the square of the first (or common) term, plus the algebraic sum of the second terms multiplied by the first (or common) term, plus the product of the second terms.*

Dem. The truth of the proposition is seen in the following results, obtained by performing in the usual way the operations indicated:

$$(x+a)(x+b)=x^2+(a+b)x+ab,$$
$$(x+a)(x-b)=x^2+(a-b)x+a(-b),$$
$$(x-a)(x+b)=x^2+(b-a)x+b(-a),$$
$$(x-a)(x-b)=x^2+(-a-b)x+(-a)(-b).$$

EXAMPLES X

Multiply the following:

1. $x+8$ by $x-5.$

Sug. Omitting the intermediate steps, and writing at once the result by the proposition, we have

$$(x+8)(x-5)=x^2+3x-40.$$

2. $x-7$ by $x+3.$ *Ans.* $x^2-4x-21.$

3. $x-15$ by $x-4.$ *Ans.* $x^2-19x+60.$

4. $a+23$ by $a+4.$ *Ans.* $a^2+27a+92.$

5. $x+10$ by $x-6.$

6. $x-12$ by $x+8.$

7. $m+20$ by $m-4.$

8. $m+16$ by $m+3.$

9. $x-11$ by $x-7.$

10. $y-22$ by $y+4.$

11. $y+90$ by $y-10.$

12. $x+50$ by $x-25.$

13. $z^2 + 9$ by $z^2 - 6$.

Operation. Here z^2 takes the place of x in the demonstration, and we have

$$(z^2 + 9)(z^2 - 6) = z^4 + 3z^2 - 54.$$

14. $x^2 - 8$ by $x^2 - 6$.

15. $x^2 - 12$ by $x^2 + 7$.

16. $x^3 + 14$ by $x^3 - 5$.

17. $z^4 - 30$ by $z^4 + 5$.

18. $ax^3 - 11$ by $ax^3 + 5$.

Operation. Here ax^3 takes the place of x in the demonstration, and we have

$$(ax^3 - 11)(ax^3 + 5) = a^2x^6 - 6ax^3 - 55.$$

19. $a^2x^2 + 6$ by $a^2x^2 - 8$. *Ans.* $a^4x^4 - 2a^2x^2 - 48$.

20. $mx^3 - 13$ by $mx^3 + 3$.

21. $m^2z^4 + 15$ by $m^2z^4 - 6$.

22. $a^3x^2 - 20$ by $a^3x^2 + 4$.

23. $a^3x^3 + 23$ by $a^3x^3 - 3$.

24. $3x^2 + 6$ by $3x^2 - 4$.

Operation. Here $3x^2$ takes the place of x in the demonstration, and we have

$$(3x^2 + 6)(3x^2 - 4) = 9x^4 + 2(3x^2) - 24 = 9x^4 + 6x^2 - 24.$$

25. $2x - 9$ by $2x + 7$.

26. $4x^2 + 11$ by $4x^2 - 8$.

27. $3z^3 - 12$ by $3z^3 + 6$.

28. $5z^2 - 15$ by $5z^2 - 5$.

29. $x + 7y$ by $x - 3y$.

Operation. Here $7y$ and $-3y$ take the places of a and b, respectively, in the demonstration, and we have

$$(x + 7y)(x - 3y) = x^2 + 4xy - 21y^2.$$

30. $x - 10y$ by $x - 6y$.

31. $x + 12y$ by $x + 4y$.

32. $x - 9y$ by $x - 7y$.

33. $x^2 + 8y$ by $x^2 - 4y$.

34. $2z^2 - 5y$ by $2z^2 + 9y$.

35. $z^2 + 4y^3$ by $z^2 - 8y^3$.

36. $z^3 - 11y^2$ by $z^3 + 7y^2$.

37. $3x^3 + 7y^2$ by $3x^3 - 2y^2$.

38. $x^2 + 5m$ by $x^2 - 2n$.

Operation. Here x^2 takes the place of x in the demonstration, and $5m$ and $-2n$ of a and b, respectively, and we have

$$(x^2 + 5m)(x^2 - 2n) = x^4 + (5m - 2n)x^2 - 10mn.$$

39. $x - 8y$ by $x + 3z$.

40. $x^2 + 11y$ by $x^2 - 4z$.

41. $3x^3 + 7y$ by $3x^3 - 5z$.

42. $5a^2 + 6m^3$ by $5a^2 - 3n^2$.

61. Theorem. *The square of the sum of two quantities is equal to the square of the first, plus twice the product of the two, plus the square of the second.*

DEM. By actual multiplication, we have

$$(x+y)^2 = (x+y)(x+y) = x^2 + 2\,xy + y^2.$$

62. Theorem. *The square of the difference of two quantities is equal to the square of the first, minus twice the product of the two, plus the square of the second.*

DEM. By actual multiplication, we have

$$(x-y)^2 = (x-y)(x-y) = x^2 - 2\,xy + y^2.$$

63. Theorem. *The product of the sum and difference of two quantities is equal to the difference of their squares.*

DEM. By actual multiplication, we have

$$(x+y)(x-y) = x^2 - y^2.$$

EXAMPLES XI

Square the following:

1. $1 + x.$ **2.** $x + 2.$ **3.** $x^2 + y.$

4. $x - 5\,y.$ **5.** $3\,a + 4\,b.$ **6.** $x^n + 2.$

7. $\dfrac{x}{y} \pm \dfrac{y}{x}.$ **8.** $\frac{2}{3}\,x^{\frac{1}{4}} - \frac{3}{4}\,x^{-\frac{1}{4}}.$ **9.** $x^{\frac{3}{4}} + \frac{1}{6}\,y^{-\frac{1}{2}}.$

10. $x^2 + y^2z^3.$ **11.** $7\,x^3 - 3\,y^2.$ **12.** $a^{\frac{1}{4}} - \frac{1}{2}\,a^{\frac{3}{4}}b^3.$

13. $x^{2n} - y^n.$ **14.** $ax^3 - by^2.$ **15.** $5\,a^3x^4 + 4\,b^2y^3.$

16. $4\,x + 7\,x^{n+1}.$ **17.** $7\,x^{\frac{3}{2}} - 3\,y^{-\frac{5}{2}}.$ **18.** $\dfrac{2\,a^m}{3\,b^n} \pm \dfrac{3\,b^n}{2\,a^m}.$

Write the products of the following:

19. $x^2 + 2\,y$ by $x^2 - 2\,y.$ **20.** $3\,m^2 + 5\,n^3$ by $3\,m^2 - 5\,n^3.$

21. $1 + \frac{2}{3}\,a$ by $1 - \frac{2}{3}\,a.$ **22.** $5\,x^3 + 3\,y^2z$ by $5\,x^3 - 3\,y^2z.$

23. $2\,x^{\frac{1}{3}} + 3\,y^{\frac{1}{2}}$ by $2\,x^{\frac{1}{3}} - 3\,y^{\frac{1}{2}}.$ **24.** $9\,ax + \sqrt{ax}$ by $9\,ax - a^{\frac{1}{2}}x^{\frac{1}{2}}.$

64. Theorem. *The square of any polynomial is the sum of the squares of each of the terms, together with the algebraic sum of twice each term into each of the terms that follow it.*

DEM. When there are more than two terms, part of them may be treated as constituting the first term and the others as the second term of a binomial. Thus:

$$(a+b+c)^2 = [(a+b)+c]^2 = (a+b)^2 + 2(a+b)c + c^2$$
$$= a^2 + 2ab + b^2 + 2ac + 2bc + c^2,$$

or
$$a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

Again,

$$(a+b+c-d)^2 = [(a+b)+(c-d)]^2 = (a+b)^2 + 2(a+b)(c-d) + (c-d)^2$$
$$= a^2 + 2ab + b^2 + 2ac - 2ad - 2bd + c^2 - 2cd + d^2,$$

or
$$a^2 + b^2 + c^2 + d^2 + 2ab + 2ac - 2ad + 2bc - 2bd - 2cd.$$

By inspecting the results of these operations we deduce the important, time-saving theorem stated above.

EXAMPLES XII

Square the following:

1. $a - b - c$.
2. $a^2 - b^2 + c$.
3. $m - n - p + q$.
4. $x^2 + 2y - u - 3v$.
5. $x^3 + x^2 + x + 1$.
6. $a - b + c - d + e$.
7. $x^2 - xy + y^2$.
8. $x^3 + x^2y + xy^2 + y^3$.
9. $3x^2 - 5y + 4z^3 - 2u$.
10. $a^4 - 2b^3 - 3c^2 + 4d$.

65. Theorem. *The product of the sum of two groups of terms by the difference of the same groups is equal to the difference of the squares of the groups.*

This is but an application of Art. 63, regarding the first group as one term and the second group as the other term of a binomial.

EXAMPLES XIII

Write the products of the following:

1. $a+b-c$ by $a-b+c$.

OPERATION. This may be written,

$\{a+(b-c)\}\{a-(b-c)\}=a^2-(b^2-2bc+c^2)=a^2-b^2+2bc-c^2.$

2. $x-3y+5z$ by $x-3y-5z$.

3. x^2+y^2+xy by x^2+y^2-xy.

4. $2x-4y-z$ by $2x+4y-z$.

5. $4x^4-x^2y^2+3y^4$ by $4x^4+x^2y^2+3y^4$.

6. x^2-x+9 by x^2-x-9.

7. x^2+x-7 by x^2-x+7.

8. $3x^2-4x+5$ by $3x^2+4x+5$.

9. $a+2b+3c+d$ by $a+2b-3c-d$.

CHAPTER V

DIVISION

66. Division is the inverse of multiplication, and is, therefore, the process of finding either factor when the other factor and the product are given.

The terms *Dividend, Divisor, Quotient, Remainder, Numerator,* and *Denominator* are used as in Arithmetic.

67. Theorem. *When dividend and divisor have like signs, the quotient is* $+$, *and when unlike,* $-$.

DEM. If d is the divisor and q the quotient, qd is the dividend (Art. 66). Now by the law of signs in multiplication (Art. 46), if d and qd have like signs, q is $+$; and if d and qd have unlike signs, q is $-$.

68. Theorem. *The exponent of a quantity in the quotient is the exponent of this quantity in the dividend, diminished by its exponent in the divisor.*

DEM. Since the exponent of a quantity in the dividend, which is the product of divisor and quotient, is the sum of the exponents of that quantity in these factors (Art. 48), then, conversely, the exponent of a quantity in the quotient (one of the factors) is the exponent of this quantity in the dividend (the product), diminished by its exponent in the divisor (the other factor).

69. Cor. 1. *Negative exponents arise from division when the exponent of a quantity in the divisor is greater than that of the same quantity in the dividend.*

70. Cor. 2. *Any quantity with the exponent 0 is 1.*

DEM. Let x represent any quantity and m any exponent. Now $x^m \div x^m = 1$. But by the law of exponents just established,

$$x^m \div x^m = x^{m-m} = x^0. \quad \text{Hence } x^0 = 1.$$

71. Cor. 3. *A factor may be transferred from dividend to divisor (or from numerator to denominator of a fraction, which is the same thing), and vice versa, by changing the sign of its exponent.*

For $$\frac{a}{b\,x^m} = \frac{ax^0}{b\,x^m} = \frac{ax^{0-m}}{b} = \frac{ax^{-m}}{b}.$$

EXAMPLES XIV

Divide the following:

1. $4\,a^5$ by $2\,a^3$.
2. $20\,ax^3$ by $4\,ax$.
3. $32\,x^5y^4z^3$ by $8\,x^3y^4z^5$.
4. $27\,x^m$ by $9\,x^n$.
5. $42\,x^m$ by $7\,x^{-n}$.
6. $a^3x^{\frac{2}{3}}$ by $ax^{\frac{1}{2}}$.
7. $ax^{\frac{m}{n}}$ by a^2x.
8. $(ab)^{2m}$ by $(ab)^{-m}$.
9. $18\,x^{-3}$ by $3\,x^2$.
10. $m^4x^{-\frac{2}{3}}$ by m^2x^{-2}.
11. $a^5(a-x)^4$ by $a^3(a-x)^2$.
12. $12\,m^{-2}(a^2-3\,x)^3$ by $3\,m^{-3}(a^2-3\,x)^{-2}$.
13. $3^7(x^2-y^2)^{\frac{3}{2}}$ by $3^4(x^2-y^2)^{\frac{1}{2}}$.
14. $a^3(2\,x+4\,x^3)^{2n-3}$ by $a(2\,x+4\,x^3)^{n-3}$.

Free the following from negative exponents:

15. $\dfrac{a^2b^{-3}}{x^{-2}y^3}$.
16. $\dfrac{3\,ab^{-2}c^{-1}x^3}{5\,a^{-2}d^{-3}x^{-2}}$.
17. $\dfrac{4\,x^my^{-n}z^{-p}}{7\,x^{-n}y^m}$.

72. Theorem. *The quotient of the sum or difference of several quantities is equal to the sum or difference of the quotients, the divisor being the same.*

Dem. Thus, $\dfrac{x+y-z}{n}$ is equal to $\dfrac{x}{n}+\dfrac{y}{n}-\dfrac{z}{n}$, for the product of each by n is the same, viz. $x+y-z$.

EXAMPLES XV

Divide the following:

1. $6\,a^2b^3+15\,a^4b^2-12\,a^2b^2$ by $3\,ab$.
2. $28\,x^5y^4-84\,x^3y^5+63\,x^3y^3$ by $7\,x^2y^3$.

3. $15\,ax^3 - 20\,a^2x^2 + 5\,a^3x$ by $-5\,ax$.

4. $40\,a^3bc - 24\,ab^3c - 32\,abc^3$ by $8\,abc$.

5. $9\,x^{2m} + 6\,x^{3m} - 12\,x^{4m}$ by $3\,x^m$.

6. $4\,a^{m-n}b^{s-n} - 6\,a^{m+n}b^{s+n}$ by $2\,a^{-n}b^n$.

7. $84\,a^{10}b^4 - 12\,a^{-10}b^8 + 156\,a^2b^6$ by $12\,a^{-10}$.

8. $x^{\frac{5}{3}} + 3\,x^{\frac{5}{3}}y^2 - 2\,x^{\frac{1}{3}}$ by $x^{\frac{1}{3}}$.

9. $a^{1+n} - a^{1+2m} - a^{1+3n} + a^{1+4n}$ by a^{3n}.

10. $6\,(x-y)^{n-2} - 9\,(x-y)^{n-1} + 12\,(x-y)^n$ by $3\,(x-y)^{n-3}$.

73. Prob. *To divide one polynomial by another.*

RULE. *Arrange dividend and divisor with reference to the same letter.*

For the first term of the quotient divide the first term of the dividend by the first term of the divisor.

Multiply the divisor by this term of the quotient, and subtract the product from the dividend.

Treat the remainder as a new dividend and proceed as before, continuing the operation until there is no remainder, or until the first term of the remainder is not divisible by the first term of the divisor.

DEM. Since the dividend is the product of the divisor and quotient (Art. 66), that term of the dividend which has the highest exponent of the letter of arrangement must be the product of those terms of the divisor and quotient which contain the highest exponents of the same letter. Hence, if we divide the first term of the arranged dividend by the first term of the arranged divisor, we shall obtain the first term of the quotient.

If the product of the whole divisor by the first term of the quotient be subtracted from the dividend, the remainder must be the product of the divisor by the sum of all the other terms of the quotient; hence the second term of the quotient may be found from this new dividend as the first was found from the original dividend.

If, finally, there is no remainder, the division is exact. If, in the end, there is a remainder whose first term is not divisible by

the first term of the divisor, the quotient may be completed, if desired, by adding this remainder over the divisor in the form of a fraction.

74. Sch. The arrangement of the terms corresponds with the succession of thousands, hundreds, etc., in decimal numbers, and these operations of division are analogous to those of long division in Arithmetic.

Example

Divide $6x^4 - 13ax^3 + 13a^2x^2 - 13a^3x - 5a^4$ by $2x^2 - 3ax - a^2$.

Operation

$$
\begin{array}{rrrrr|l}
6x^4 & -\ 13ax^3 & +\ 13a^2x^2 & -\ 13a^3x & -\ 5a^4 & 2x^2 - 3ax - a^2 \\ \cline{6-6}
6x^4 & -\ 9ax^3 & -\ 3a^2x^2 & & & 3x^2 - 2ax + 5a^2 \\ \cline{1-3}
 & -\ 4ax^3 & +\ 16a^2x^2 & -\ 13a^3x & & \\
 & -\ 4ax^3 & +\ 6a^2x^2 & +\ 2a^3x & & \\ \cline{2-5}
 & & 10a^2x^2 & -\ 15a^3x & -\ 5a^4 & \\
 & & 10a^2x^2 & -\ 15a^3x & -\ 5a^4 & \\ \cline{3-5}
\end{array}
$$

75. A simple inspection will show that the following may be omitted from the operation as entirely unnecessary:

1. The first term of each product to be subtracted, because it is always the same as the first term of that from which it is to be subtracted.

2. The literal factors in the various terms of the products and remainders, because they are always the same as those of the terms of the dividend under which they stand.

3. The bringing down of terms from the dividend to the remainders, because the subtractions can just as well be made while they retain their original positions.

4. The signs except when $-$, because when no sign is written, the $+$ sign is always understood.

With these omissions the above operation contracts to the following:

$$
\begin{array}{rrrrr|l}
6x^4 & -\ 13ax^3 & +\ 13a^2x^2 & -\ 13a^3x & -\ 5a^4 & 2x^2 - 3ax - a^2 \\ \cline{6-6}
 & -\ 9 & -\ 3 & & & 3x^2 - 2ax + 5a^2 \\ \cline{1-3}
 & -\ 4 & 16 & & & \\
 & & 6 & 2 & & \\ \cline{3-4}
 & & 10 & -\ 15 & & \\
 & & & -\ 15 & -\ 5 & \\ \cline{4-5}
\end{array}
$$

When terms are missing from the dividend, their places must be supplied by terms with coefficient zero.

EXAMPLES XVI

Divide the following:

1. $1+2x^2-7x^4-16x^6$ by $1+2x+3x^2+4x^3$.

OPERATION

$$\begin{array}{rrrrrrr|l}
1+ & 0x+ & 2x^2+ & 0x^3- & 7x^4+ & 0x^5- & 16x^6 & 1+2x+3x^2+4x^3 \\ \cline{8-8}
 & 2 & 3 & 4 & & & & \\ \cline{2-4}
 & -2 & -1 & -4 & & & & \\
 & & -4 & -6 & -8 & & & \\ \cline{3-5}
 & & 3 & 2 & 1 & & & \\
 & & & 6 & 9 & 12 & & \\ \cline{4-6}
 & & & -4 & -8 & -12 & & \\
 & & & & -8 & -12 & -16 & \\ \cline{5-7}
\end{array}$$

2. $15x^4-32x^3+50x^2-32x+15$ by $3x^2-4x+5$.

3. $6x^4-31x^3+23x^2-2x-48$ by $3x^2-5x+6$.

4. $2a^7b-5a^6b^2-11a^5b^3+5a^4b^4-26a^3b^5+7a^2b^6-12ab^7$ by $a^4-4a^3b+a^2b^2-3ab^3$.

SHORT METHODS OF DIVISION

76. The operation of division may be still further shortened as follows:

If the signs of the terms of the divisor after the first are changed, the signs of those terms of the various products which were subtracted in the operations above will be changed, and the subtraction changed to addition, since to subtract we change the signs of the subtrahend and proceed as in addition. The operation at the bottom of page 48 would then become,

$$\begin{array}{rrrrr|l}
6x^4- & 13ax^3+ & 13a^2x^2- & 13a^3x- & 5a^4 & 2x^2+3ax+a^2 \\ \cline{6-6}
 & 9 & 3 & & & 3x^2-2ax+5a^2 \\ \cline{2-3}
 & -4 & 16 & & & \\
 & & -6 & -2 & & \\ \cline{3-4}
 & & 10 & -15 & & \\
 & & & 15 & 5 & \\ \cline{4-5}
\end{array}$$

It will be seen that the result of the first addition to the term $13\,a^2x^2$ is not used in obtaining a term of the quotient, and that the result which is used, viz., 10 (with a^2x^2 understood), can just as well be obtained by adding 13, 3, and -6 without first adding 13 and 3. The same remarks apply to the additions to the term $-13\,a^3x$. With these omissions, and writing the terms of the products as near as may be to the corresponding terms of the dividend, the operation becomes,

$$
\begin{array}{ccccc|l}
6\,x^4 & -\,13\,ax^3 & +\,13\,a^2x^2 & -\,13\,a^3x & -\,5\,a^4 & \;2\,x^2+3\,ax+a^2 \\ \cline{6-6}
 & \underline{\;9\;} & 3 & -\,2 & 5 & \;3\,x^2-2\,ax+5\,a^2 \\
 & -4 & \underline{-6} & \underline{15} & \underline{\quad} & \\
 & & 10 & 0 & 0 &
\end{array}
$$

A convenient rule for this operation is the following:

Arrange dividend and divisor with reference to the same letter, supplying the places of any missing terms of the dividend by terms with zero coefficients.

Change the signs of all the terms of the divisor except the first.

Divide the first term of the dividend by the first term of the divisor for the first term of the quotient.

Multiply the terms of the changed divisor except the first by this term of the quotient and write the product under the corresponding terms of the dividend, omitting the literal factors.

Add the first of these products to the second term of the dividend, and divide this sum by the first term of the divisor for the second term of the quotient.

Multiply the terms of the changed divisor except the first by this second term of the quotient and write the products under the corresponding terms of the dividend, omitting the literal factors.

Add the third term of the dividend to the two terms that stand under it, and divide this sum by the first term of the divisor for the third term of the quotient.

Continue this operation either until the division terminates or until one of these sums contains a lower exponent of the letter of arrangement than the first term of the divisor. In the latter case this sum and the sums of the remaining terms of the dividend and the terms that stand under them constitute the remainder.

77. This is a modified form of **Horner's Synthetic Division** and should be thoroughly mastered by the student. It requires but little attention and practice to become familiar with it, while the saving in time is very great. By comparing the last operation with the usual process as given in the first solution it will be seen that while that operation (not counting dividend, divisor, and quotient) contains 80 figures, letters, and signs, this contains but 15.

EXAMPLES XVII

Divide the following:

1. $10\,a^4 - 27\,a^3x + 34\,a^2x^2 - 18\,ax^3 - 8\,x^4$ by $2\,a^2 - 3\,ax + 4\,x^2$.

2. $x^4 - 3\,ax^3 - 8\,a^2x^2 + 18\,a^3x - 8\,a^4$ by $x^2 + 2\,ax - 2\,a^2$.

3. $4\,y^6 - 24\,y^5 + 60\,y^4 - 80\,y^3 + 60\,y^2 - 24\,y + 4$ by $2\,y^2 - 4\,y + 2$.

4. $x^6 - 5\,x^5 + 15\,x^4 - 24\,x^3 + 27\,x^2 - 13\,x + 5$ by $x^4 - 2\,x^3 + 4\,x^2 - 2\,x + 1$.

5. $2\,a^7b - 5\,a^6b^2 - 11\,a^5b^3 + 5\,a^4b^4 - 26\,a^3b^5 + 7\,a^2b^6 - 12\,ab^7$ by $a^4 - 4\,a^3b + a^2b^2 - 3\,ab^3$.

6. $a^6 - 3\,a^4x^2 + 3\,a^2x^4 - x^6$ by $a^3 - 3\,a^2x + 3\,ax^2 - x^3$.

7. $9\,x^2 + 24\,xy + 12\,y^2 + 30\,xz + 24\,yz + 9\,z^2$ by $x + 2\,y + 3\,z$.

8. $8\,y^5 - 22\,xy^4 + 20\,x^2y^3 + x^3y^2 - 7\,x^4y + 6\,x^5$ by $4\,y^2 - 3\,xy + 2\,x^2$.

9. $6\,x^5 - 7\,x^4y + x^3y^2 + 20\,x^2y^3 - 22\,xy^4 + 8\,y^5$ by $2\,x^2 - 3\,xy + 4\,y^2$.

10. $1 + 2\,x^5 + x^6 + 2\,x^7$ by $1 + x + x^2$.

11. $16\,x^4 + 36\,x^2 + 81$ by $4\,x^2 - 6\,x + 9$.

12. $6\,x - 20 - 2\,x^5 + 15\,x^2 - x^4 + 2\,x^3$ by $5 + 2\,x^3 - 4\,x - 3\,x^2$.

13. $12\,a^5 - 14\,a^4b - 10\,a^3b^2 - a^2b^3 - 8\,ab^4 + 4\,b^5$ by $6\,a^3 - 4\,a^2b - 3\,ab^2 + 2\,b^3$.

14. $14\,x^4 + 45\,x^3y + 78\,x^2y^2 + 45\,xy^3 + 21\,y^4$ by $2\,x^2 + 5\,xy + 7\,y^2$.

15. $1 - x^6$ by $1 + 2\,x + 2\,x^2 + x^3$.

OPERATION

$$\begin{array}{ccccccc|l}
1 + & 0\,x + & 0\,x^2 + & 0\,x^3 + & 0\,x^4 + & 0\,x^5 & - x^6 & 1 - 2\,x - 2\,x^2 - x^3 \\
\hline
 & \underline{-2} & -2 & -1 & 2 & -2 & 1 & 1 - 2\,x + 2\,x^2 - x^3 \\
 & -2 & \underline{4} & 4 & -4 & 2 & & \\
 & & 2 & \underline{-4} & 2 & & & \\
 & & & -1 & 0 & 0 & 0 &
\end{array}$$

16. $x^5 + y^5$ by $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.

17. $a^8 + a^6b^2 + a^4b^4 + a^2b^6 + b^8$ by $a^4 + a^3b + a^2b^2 + ab^3 + b^4$.

18. $x^6 - 21\,x^3y^3 + 24\,xy^5 - 8\,y^6$ by $x^2 - 3\,xy + y^2$.

SUG. From the terms given, the arrangement is seen to be with reference to the descending powers of one letter and the ascending powers of the other. The dividend should, therefore, be written

$$x^6 + 0\,x^5y + 0\,x^4y^2 - 21\,x^3y^3 + 0\,x^2y^4 + 24\,xy^5 - 8\,y^6.$$

19. $\frac{9}{16}\,a^4 - \frac{3}{4}\,a^3 - \frac{7}{4}\,a^2 + \frac{4}{3}\,a + \frac{16}{9}$ by $\frac{3}{2}\,a^2 - a - \frac{8}{3}$.

20. $36\,x^2 - 6\,x - 4\,xy + \frac{1}{3}\,y + \frac{1}{9}\,y^2 + \frac{1}{4}$ by $6\,x - \frac{1}{3}\,y - \frac{1}{2}$.

21. $abx^4 + 2\,(a - b)\,x^3 - (a^2 + 4 - b^2)\,x^2 + 2\,(a + b)\,x - ab$ by $bx^2 + 2\,x - a$.

22. $6\,x^4 - 25\,x^3y + 16\,x^2y^2 - 17\,xy^3 + 35\,y^4$ by $2\,x - 7\,y$.

SUG. When the divisor is a binomial, as in this example, the operation takes the following form:

$$\begin{array}{ccccc|l}
6\,x^4 & -\,25\,x^3y & +\,16\,x^2y^2 & -\,17\,xy^3 & +\,35\,y^4 & 2\,x\ +7\,y \\ \cline{6-6}
 & 21 & -\,14 & 7 & -\,35 & 3\,x^3 - 2\,x^2y + xy^2 - 5\,y^3 \\ \cline{2-5}
 & -\,4 & 2 & -\,10 & 0 &
\end{array}$$

23. $x + 9\,x^3 - 1 + 3\,x^2$ by $3\,x - 1$.

24. $7\,x^3 - 24\,x^2 + 58\,x - 21$ by $7\,x - 3$.

25. $2\,a^4 + 27\,ab^3 - 81\,b^4$ by $a + 3\,b$.

26. $6\,x^5 - 11\,x^3 + 3\,x^2 + 4\,x - 4$ by $3\,x^2 - 4$.

27. $6\,a^4 - 96$ by $3\,a - 6$.

28. $a^{1+n} + a^nb + ab^n + b^{1+n}$ by $a^n + b^n$.

29. $m^{m+1} + nm^m + amn^{an} + an^{an+1}$ by $m + n$.

30. $6\,a^4 - 13\,a^3b + 9\,a^2b^2 + 10\,ab^3 - 8\,b^4$ by $3\,a - 2\,b$.

31. $10\,x^5 - 7\,x^4y + 8\,x^3y^2 - 9\,x^2y^3 + 10\,xy^4 + 28\,y^5$ by $5\,x + 4\,y$.

78. *When the coefficient of the first term of a binomial divisor of the first degree is unity,* the operation becomes exceedingly simple.

The process is of so much importance and is of such frequent occurrence in subsequent parts of the work that a special rule is here given.

79. Prob. *To divide by a binomial of the first degree when the coefficient of the first term is unity.*

RULE. *Call the coefficient of the second term of the divisor, with its sign changed, a; multiply the coefficient of the first term of the dividend by a, and add the product to that of the second; multiply this sum by a, and add the product to the coefficient of the third term of the dividend, and so on. When the last sum is zero, the division is exact; otherwise, the last sum, with the literal part of the last term of the dividend, will be the remainder, and the coefficient of the first term of the dividend, and the other sums in order, will be the coefficients of the quotient, the arrangement of the letters of the quotient being the same as in the dividend, and the degree being one less than that of the dividend.*

NOTE. This rule may be deduced as a special case of the last process; but it is deduced below directly from the ordinary process of "long division," with which the student has become familiar in Arithmetic and Elementary Algebra, in order that the relation to that process, and at the same time the great saving in labor and time, may be seen.

DEM. By the ordinary process of "long division" we have,

$$\begin{array}{llllll|l}
2x^5 & -3x^4y & -13x^3y^2 & +14x^2y^3 & -2xy^4 & -12y^5 & x-3y \\
\cline{7-7}
2x^5 & -6x^4y & & & & & 2x^4+3x^3y-4x^2y^2 \\
\cline{1-2}
 & 3x^4y & -13x^3y^2 & & & & +2xy^3+4y^4 \\
 & 3x^4y & -9x^3y^2 & & & & \\
\cline{2-3}
 & & -4x^3y^2 & +14x^2y^3 & & & \\
 & & -4x^3y^2 & +12x^2y^3 & & & \\
\cline{3-4}
 & & & 2x^2y^3 & -2xy^4 & & \\
 & & & 2x^2y^3 & -6xy^4 & & \\
\cline{4-5}
 & & & & 4xy^4 & -12y^5 & \\
 & & & & 4xy^4 & -12y^5 & \\
\cline{5-6}
\end{array}$$

Omitting the wholly unnecessary parts, without changing the position of the parts that remain, this becomes,

$$
\begin{array}{r|r|r|r|r|l}
2x^5 - 3 & x^4y - 13 & x^3y^2 + 14 & x^2y^3 - 2 & xy^4 - 12 & y^5 \;\underline{\big|\, x - 3y} \\
\underline{-6} & & & & & 2x^4 + 3x^3y - 4x^2y^2 \\
3 & & & & & \quad + 2xy^3 + 4y^4 \\
 & \underline{-9} & & & & \\
 & -4 & & & & \\
 & & \underline{12} & & & \\
 & & 2 & & & \\
 & & & \underline{-6} & & \\
 & & & 4 & & \\
 & & & & \underline{-12} & \\
 & & & & 0 &
\end{array}
$$

Now since the coefficient of the first term of the divisor is, by hypothesis, unity, the coefficients of the terms of the quotient will be the same as the coefficients of those terms which are divided by x to obtain them (in this case the same as in the terms $2x^5$, $3x^4y$, $-4x^3y^2$, $2x^2y^3$, and $4xy^4$). Hence the coefficient of the first term of the dividend and the coefficients of the various remainders in order will be the coefficients of the terms of the quotient. Any one of these remainders is found by multiplying the one before it (which is the same as the coefficient of the corresponding term of the quotient) by the coefficient of the second term of the divisor, and subtracting the product from the coefficient of the next term of the dividend. By changing the sign of the second term of the divisor this subtraction will be changed to addition. The degree of the quotient is one less than that of the dividend, because the divisor is of the first degree. Hence writing in place of the divisor only the coefficient of the second term with its sign changed, omitting from its usual place the quotient, since its coefficients are but a repetition of numbers written before, and writing the products and sums as near as may be to the terms of the dividend with which they belong, the operation becomes,

$$
\begin{array}{cccccc}
2x^5 - & 3x^4y - & 13x^3y^2 + & 14x^2y^3 - & 2xy^4 - & 12y^5 \;\underline{\big|\, 3} \\
 & \underline{6} & \underline{9} & \underline{-12} & \underline{6} & \underline{12} \\
 & 3 & -4 & 2 & 4 & 0
\end{array}
$$

Supplying now the literal parts, we have for the quotient,

$$2x^4 + 3x^3y - 4x^2y^2 + 2xy^3 + 4y^4.$$

When the coefficients are small, as in this case, or so related that the sums are small, the additions can be readily made without writing the products.

$$\begin{array}{cccccc|l} 2x^5 - & 3x^4y - & 13x^3y^2 + & 14x^2y^3 - & 2xy^4 - & 12y^5 & \underline{3} \\ & 3 & -4 & 2 & 4 & 0 & \end{array}$$

Thus, 3 times 2 are 6, which added to -3 gives 3; 3 times 3 are 9, which added to -13 gives -4; 3 times -4 are -12, which added to 14 gives 2; 3 times 2 are 6, which added to -2 gives 4; 3 times 4 are 12, which added to -12 gives 0. Supplying the literal parts, we have for the quotient,

$$2x^4 + 3x^3y - 4x^2y^2 + 2xy^3 + 4y^4.$$

When the last sum is anything other than zero, this sum (including the literal part) is the remainder, since the operation by which it is obtained is the same as subtracting the product of the last terms of the divisor and quotient from the last term of the dividend.

80. By comparing the last operation with the usual process by "long division," as given in the first operation, it will be seen that while that operation (not counting dividend, divisor, and quotient) contains 93 figures, letters, and signs, this contains but 6.

EXAMPLES XVIII

Divide the following:

1. $x^3 - 11x^2y + 41xy^2 - 50y^3$ by $x - 5y$.

Operation

$$\begin{array}{cccc|l} x^3 - & 11x^2y + & 41xy^2 - & 50y^3 & \underline{5} \\ & -6 & 11 & 5 & \end{array}$$

1, -6, and 11 are the coefficients of the terms of the quotient and 5 is the coefficient of the remainder.

Ans. $x^2 - 6xy + 11y^2$, with remainder $5y^3$.

2. $7x^3 - 3x^2y - 2xy^2 - 40y^3$ by $x - 2y$.

3. $x^4 + 4x^3y - 13x^2y^2 - 28xy^3 + 60y^4$ by $x - 2y$.

4. $2x^4 + 11x^3y + 9x^2y^2 - 8xy^3 + 16y^4$ by $x + 4y$.

5. $12x^4 - 48x^3y + 11x^2y^2 - 45xy^3 + 7y^4$ by $x - 4y$.

6. $x^5 + 4x^4y - 2x^3y^2 - 18x^2y^3 - 6xy^4 + 9y^5$ by $x + 3y$.

7. $3x^5 + 2x^4y - 21x^3y^2 - 14x^2y^3 + 36xy^4 + 24y^5$ by $x - 2y$.

8. $3x^5 - 25x^4y - 15x^3y^2 - 31x^2y^3 + 38xy^4 - 18y^5$ by $x - 9y$.

9. $4x^6 - 24x^5y + 23x^4y^2 - 14x^3y^3 + x^2y^4 - 25xy^5 - 25y^6$ by $x - 5y$.

10. $x^6 - 2x^5y - 45x^4y^2 + 22x^3y^3 + 24x^2y^4 + 2xy^5 + 12y^6$ by $x + 6y$.

11. $2x^5 - 9x^4 + 11x^3 - 20x + 6$ by $x - 3$.

OPERATION

$$\begin{array}{cccccc|l} 2x^5 & -9x^4 & +11x^3 & +0x^2 & -20x & +6 & \underline{3} \\ & -3 & 2 & 6 & -2 & 0 & \end{array}$$

Ans. $2x^4 - 3x^3 + 2x^2 + 6x - 2$.

12. $x^4 + x^3 - 50x^2 - 40x - 14$ by $x - 7$.

13. $3x^5 + 85x^2 - 3x - 45$ by $x + 3$.

14. $8x^3 - 8x^2 - 60x + 400$ by $x + 4$.

15. $5x^4 - 46x^3 + 48x^2 + 7x - 56$ by $x - 8$.

16. $x^5 - 20x^3 - 20x^2 - 20x - 25$ by $x - 5$.

17. $x^5 - 13x^4 + 15x^3 - 30x^2 - 75x + 36$ by $x - 12$.

18. $2x^4 + 35x^3 + 80x^2 + 75x$ by $x + 15$.

19. $x^6 + 3x^5 - 7x^4 - 10x^3 + 6x^2 + 9x - 10$ by $x - 2$.

20. $x^5 - 25x^4 + 47x^3 - 21x^2 - 43x - 69$ by $x - 23$.

21. $x^6 - 64$ by $x - 2$.

OPERATION

$$\begin{array}{ccccccc|l} x^6 & +0x^5 & +0x^4 & +0x^3 & +0x^2 & +0x & -64 & \underline{2} \\ & 2 & 4 & 8 & 16 & 32 & 0 & \end{array}$$

Ans. $x^5 + 2x^4 + 4x^3 + 8x^2 + 16x + 32$.

22. $x^4 - 81$ by $x - 3$.

23. $x^5 + y^5$ by $x + y$.

24. $x^6 - 729y^6$ by $x - 3y$.

81. Successive Division. In dividing successively by several binomials of the first degree when the coefficient of the first term of each is unity, the letters should not be written until after the last division. It must be noted that the degree is reduced by one for each division.

EXAMPLES XIX

Divide successively the following:

1. $x^5 - 3x^4 - 9x^3 + 21x^2 - 10x + 24$ by $x - 2$, $x + 3$, and $x - 4$.

OPERATION

$$\begin{array}{cccccc|c} x^5 - 3x^4 & - & 9x^3 + 21x^2 & - 10x & + 24 & & \underline{2} \\ -1 & -11 & -1 & -12 & 0 & & \underline{-3} \\ -4 & 1 & -4 & 0 & & & \underline{4} \\ 0 & 1 & 0 & & & & \end{array}$$

Ans. $x^2 + 0x + 1$, or $x^2 + 1$.

2. $x^4 - x^3 - 19x^2 + 49x - 30$ by $x - 3$ and $x + 5$.
3. $x^4 + 5x^3 - 30x^2 - 80x + 224$ by $x - 2$, $x - 4$, and $x + 4$.
4. $x^5 + x^4 - 31x^3 + 68x^2 - 54x + 63$ by $x - 3$ and $x + 7$.
5. $x^5 - 7x^4 + 5x^3 + 55x^2 - 126x + 72$ by $x - 1$, $x - 2$, $x - 3$, $x + 3$, and $x - 4$.
6. $x^6 - 14x^4 + 49x^2 - 36$ by $x - 1$, $x + 1$, $x - 2$, $x + 2$, and $x - 3$.

OPERATION

$$\begin{array}{ccccccc|c} x^6 + 0x^5 & - 14x^4 + & 0x^3 + & 49x^2 + & 0x & - 36 & & \underline{1} \\ 1 & -13 & -13 & 36 & 36 & 0 & & \underline{-1} \\ 0 & -13 & 0 & 36 & 0 & & & \underline{2} \\ 2 & -9 & -18 & 0 & & & & \underline{-2} \\ 0 & -9 & 0 & & & & & \underline{3} \\ 3 & 0 & & & & & & \end{array}$$

Ans. $x + 3$.

7. $x^5 - 10x^4 - 3x^3 + 258x^2 - 550x - 200$ successively by $x - 4$, $x - 5$, and $x + 5$.

82. Theorem. *When the dividend is the square of the first term of the divisor, plus or minus twice the product of the first and second terms, plus the square of the second, the quotient is the same as the divisor.*

This is a consequence of Arts. 61 and 62.

83. Theorem. *When the divisor is the sum or difference of two quantities and the dividend is the difference of the squares of these quantities, the quotient is the difference or sum of the same quantities.*

This is a consequence of Art. 63.

EXAMPLES XX

Divide the following:

1. $36x^4 + 36x^2y^3 + 9y^6$ by $6x^2 + 3y^3$.
2. $49a^8 - 70a^4b^5 + 25b^{10}$ by $7a^4 - 5b^5$.
3. $16a^6 - 64b^4$ by $4a^3 - 8b^2$.
4. $81x^5 - 4y^2$ by $9x^{\frac{5}{2}} + 2y$.
5. $121x^6 - 264x^3y^6 + 144y^{12}$ by $11x^3 - 12y^6$.
6. $64a^4b^2 + 112a^3bc^3 + 49a^2c^6$ by $8a^2b + 7ac^3$.

84. *When the divisor is readily separated into binomial factors of the first degree having unity for the coefficient of the first term of each,* the division is most expeditiously made by dividing successively by the factors, as in the process of Art. 81.

EXAMPLES XXI

Divide the following:

1. $x^5 + 3x^4 - 20x^3 - 60x^2 + 64x + 192$ by $x^2 - 6x + 8$.

OPERATIONS

The factors of $x^2 - 6x + 8$ being $x - 2$ and $x - 4$, we divide successively by these factors, thus:

$$\begin{array}{cccccc|l} x^5 + 3x^4 & -20x^3 & -60x^2 & +64x & +192 & & \underline{2} \\ 5 & -10 & -80 & -96 & & 0 & \underline{4} \\ 9 & 26 & 24 & 0 & & & \end{array}$$

Ans. $x^3 + 9x^2 + 26x + 24$.

2. $x^4 - 3x^3 - 7x^2 + 15x + 18$ by $x^2 - 6x + 9$.
3. $x^4 + 3x^3 - 7x^2 - 27x - 18$ by $x^2 - 9$.
4. $x^5 + 3x^4 - 17x^3 - 27x^2 + 52x + 60$ by $x^2 + 2x - 15$.
5. $x^6 - 6x^4 - 4x^3 + 9x^2 + 12x + 4$ by $x^2 - 4x + 4$.
6. $x^5 - 7x^4 + 17x^3 - 28x^2 + 37x - 20$ by $x^2 - 5x + 4$.
7. $3x^5 + x^4 - 5x^3 - 6x^2 - 17x - 14$ by $x^2 - x - 2$.

CHAPTER VI

FACTORING

85. A Factor of a quantity is a quantity that will divide it without a remainder.

86. The Factors of a quantity are those quantities which multiplied together produce it.

87. To Factor a quantity is to separate it into its factors.

88. A Prime Quantity is one which has no integral factors except itself and unity.

89. A Composite Quantity is one which has integral factors other than itself and unity.

90. Theorem. *Any monomial factor which occurs in every term of a polynomial can be removed by dividing each term of the polynomial by it.*

Thus, $$a^3x + a^2x^2 - a^5x^3 = a^2x(a + x - a^3x^2).$$

91. Theorem. *If two terms of a trinomial are positive, and the remaining term is $\pm$ twice the product of their square roots, the trinomial is the square of the sum or difference of these square roots.*

This is a consequence of Arts. 61 and 62.

92. Theorem. *The difference between two quantities is equal to the product of the sum and difference of their square roots.*

This is a consequence of Art. 63.

93. Theorem. *If a polynomial of six terms consists of three perfect squares and three double products of their square roots, taken in pairs, the polynomial is the square of the algebraic sum of these square roots.*

This is a consequence of Art. 64.

Arts. 169 and 170 furnish the best means of factoring polynomials that are perfect squares.

EXAMPLES XXII

Factor the following:

1. $a^4b + 3a^6b^4 + 4a^3b^3$.

2. $3a^4 - 6a^5 - 12a^2 + 9a^3$.

3. $12a^4b^7 - 20a^3b^8 + 16ab^7 - 8a^5b^3$.

4. $10x^{\frac{3}{2}}y^{\frac{8}{3}} + 5x^{\frac{5}{2}}y^{\frac{2}{3}} - 15x^{\frac{3}{2}}y^{\frac{5}{3}} - 20x^{\frac{1}{2}}y^{\frac{5}{3}}$.

5. $a^2 + 6ab + 9b^2$.

6. $a^4 - 8a^2b + 16b^2$.

7. $x^6 - x^3y + \frac{1}{4}y^2$.

8. $x^4 - 4y^2$.

9. $3x^2 - 48y^4$.

10. $8x^5 - 18xy^4$.

11. $x^4 - y^8$.

12. $81x^4y^2 - 1$.

13. $1 - 10xy^2 + 25x^2y^4$.

14. $4m^2x^4 + 4mnx^2y + n^2y^2$.

15. $(x+y)^2 + 2(x+y) + 1$.

16. $x^2 - (x-y)^2$.

17. $a^2 - b^2 - c^2 + 2bc$.

18. $(x+y)^4 + 4(x+y)^2 + 4$.

19. $(x^2+y)^2 - 6z^2(x^2+y) + 9z^2$.

20. $(2x^2 + 3x - 8)^2 - (x^2 - 3x - 8)^2$.

21. $a^2 + b^2 + c^2 - 2ab + 2ac - 2bc$.

22. $x^2 + y^2 + 9 + 2xy - 6x - 6y$.

23. $4x^2 - 4xy + y^2 + 12xz - 6yz + 9z^2$.

24. $9x^4 + 4y^2 + 16z^2 - 12x^2y - 24x^2z + 16yz$.

25. $16x^6 + 24x^3y^2 + 9y^4 - 40x^3z - 30y^2z + 25z^2$.

26. One factor of $x^4 - 7x^3 + 6x^2 + 23x - 15$ being $x - 5$, what is the other?

Sug. Divide as in Art. 79.

27. One factor of $x^3 - 6x^2y + 15xy^2 - 18y^3$ being $x - 3y$, what is the other?

28. Two factors of $2x^4 + 5x^3 - 13x^2 + 16$ being $x + 4$ and $x + 1$, what is the third?

29. One factor of $x^3 - 4x^2 - 35x + 150$ being $x + 6$, what are the other two?

30. One factor of $4x^3 - 32x^2 + 69x - 45$ being $x - 5$, what are the other two?

94. Prob. *To factor a trinomial of the form* $x^{2m} + px^m + q$.

SOLUTION. We have seen (Art. 60) that

$$(x^m + a)(x^m + b) = x^{2m} + (a + b)x^m + ab.$$

Hence if in an expression of the form $x^{2m} + px^m + q$, q can be resolved into two factors, as a and b, such that their sum, $a + b$, is equal to p, then the factors are $x^m + a$ and $x^m + b$.

EXAMPLES XXIII

Factor the following:

1. $x^2 + 7x + 10$.

Here $7x$ is the product of the square root of the first term, and the sum of two factors, 2 and 5, of the last term. Hence the factors of $x^2 + 7x + 10$ are $x + 2$ and $x + 5$.

2. $x^4 + 4x^2 - 21$.

Here $4x^2$ is the product of the square root of the first term, and the sum of two factors, 7 and -3, of the last term. Hence the factors of x^4+4x^2-21 are $x + 7$ and $x - 3$.

3. $x^2 - 4x - 32$.

4. $x^2 - 8x + 15$.

5. $x^2 + 10x + 9$.

6. $x^2 + 6x - 72$.

7. $x^2 + 12x - 45$.

8. $x^2 - 16x - 80$.

9. $x^2 + 13x + 30$.

10. $x^2 - 6x - 55$.

11. $x^4 + 5x^2 - 14$.

12. $x^4 - 17x^2 + 72$.

13. $x^4 + 10x^2 + 24$.

14. $x^4 + 6x^2 - 27$.

15. $x^6 - 5x^3 - 84$.

16. $x^8 + 8x^4 - 9$.

17. $(x + y)^2 + 2(x + y) - 35$.

18. $(x^2+3y)^2+13(x^2+3y)+42$.

19. $a^2 + 4ab - 12b^2$.

Here $4ab$ is the product of the square root of the first term, and the sum of two factors, $6b$ and $-2b$, of the last term. Hence the factors of $a^2 + 4ab - 12b^2$ are $a + 6b$ and $a - 2b$.

20. $x^2 - 9xy - 36y^2$.

21. $x^4 + 11x^2y^2 + 28y^4$.

22. $x^6 - 9x^3y^2 + 18y^4$.

23. $x^4 + 25x^2y + 100y^2$.

24. $x^2y^2 + 3xyz^2 - 28z^4$.

25. $x^4 - x^2y^2z^2 - 56y^4z^4$.

95. Prob. *To factor a trinomial of the form* $ax^{2m} + bx^m + c$.

SOLUTION. $ax^{2m} + bx^m + c = \frac{1}{a}[(ax^m)^2 + b(ax^m) + ac]$, since dividing and multiplying by the same quantity does not change the value of the expression. The part within the brackets may now be factored by the process of the last article if b is the sum of two factors of ac.

As the part within the brackets is to be divided by a, we may divide one of its factors by the whole of a, or one of them by one factor of a, and the other by the remaining factor of a, and thus have two integral binomial factors of the original trinomial.

EXAMPLES XXIV

Factor the following:

1. $6x^2 + 7x - 20$.

OPERATION. $6x^2 + 7x - 20 = \frac{1}{6}[(6x)^2 + 7(6x) - 120]$.

Here $7(6x)$ is the product of the square root of the first term and the sum of two factors, 15 and -8, of the last term. Hence the factors of the part within the brackets are $6x + 15$ and $6x - 8$. Dividing the first of these by 3, one of the factors of 6, and the other by 2, the remaining factor of 6, we have for the factors of the original trinomial $2x + 5$ and $3x - 4$.

2. $3x^2 - 2x - 5$.

OPERATION. $3x^2 - 2x - 5 = \frac{1}{3}[(3x)^2 - 2(3x) - 15]$
$= \frac{1}{3}(3x - 5)(3x + 3) = (3x - 5)(x + 1)$.

3. $12x^2 - 5x - 2$.

OPERATION. $12x^2 - 5x - 2 = \frac{1}{12}[(12x)^2 - 5(12x) - 24]$
$= \frac{1}{12}(12x - 8)(12x + 3) = (3x - 2)(4x + 1)$.

4. $6x^2 + 5x - 4$.

5. $10x^2 - 11x + 3$.

6. $8x^2 + 26x + 21$.

7. $6x^2 + 13x - 15$.

8. $12x^2 - 25x + 12$.

9. $15x^2 + 14x - 8$.

10. $6x^2 - 7xy - 3y^2$.

OPERATION. $6x^2 - 7xy - 3y^2 = \frac{1}{6}[(6x)^2 - 7y(6x) - 18y^2]$
$= \frac{1}{6}(6x - 9y)(6x + 2y) = (2x - 3y)(3x + y)$

11. $6x^2 - 19xy + 10y^2$.

12. $10x^2 - 3xy - 18y^2$.

13. $7a^4 + 25a^2 + 12$.

14. $14a^4 + 25a^2 + 6$.

15. $6a^4 + 7a^2b^2 - 3b^4$.

16. $8a^6 + 14a^3b^2 - 15b^4$.

17. $15x^6 - 23x^3y^2 - 28y^4$.

18. $5x^8 + 11x^4 - 12$.

19. $8x^8 - 6x^4 - 35$.

20. $10x^{10} - 31x^5y^3 + 24y^6$.

96. The methods of Arts. 94 and 95 will give the factors of any trinomial of the form $ax^{2m} + bx^my^n + cy^{2n}$ when the factors are rational. The finding of such factors when irrational or imaginary requires the solution of a quadratic equation.

97. *A polynomial having more than three terms may sometimes be factored by first removing monomial factors from two or more groups of terms.*

Thus, let it be required to factor $am + an + bm + bn$.

$$am + an + bm + bn = a(m + n) + b(m + n).$$

We now see that the binomial factor $m + n$ is common to the two terms, being contained a times in the first and b times in the second. Hence the factors are $m + n$ and $a + b$.

98. *When a part of a polynomial can be factored by any of the preceding methods, we may sometimes find the third terms of trinomial factors as follows:*

Let it be required to find the factors of

$$6x^2 - 5xy - 2x + 43y - 21y^2 - 20.$$

This has the form of the product of two trinomial factors, each having terms in x and y, and a term containing neither.

The factors of the part $6x^2 - 5xy - 21y^2$ are, by Art. 95, $3x - 7y$ and $2x + 3y$.

The product of the terms not containing x and y is -20. Hence -20 must be resolved into two factors such that the sum of the products obtained by multiplying one of these factors by $3x$ and the other by $2x$ shall be $-2x$; also the sum of the products obtained by multiplying one of these factors by $-7y$ and the other by $3y$ shall be $43y$. By trial these are found to be 5 and -4. Hence

$$6x^2 - 5xy - 2x + 43y - 21y^2 - 20 = (3x - 7y + 5)(2x + 3y - 4).$$

EXAMPLES XXV

Factor the following:

1. $ax^3 - 2ax^2 - 3x + 6$.

OPERATION. $ax^3-2ax^2-3x+6=ax^2(x-2)-3(x-2)=(x-2)(ax^2-3)$.

The result may just as well be obtained by placing the first and third terms in one group, and the second and fourth in the other.

2. $x^2 - mx - nx + mn$.
3. $a^3x + ax + a^3y + ay$.
4. $5x^3 - 20ax^2 - 2x + 8a$.
5. $28x^2 - 21xy + 24x - 18y$.
6. $x^4 - x^3y + xz^2 - yz^2$.
7. $abx^2 + bxy - axy - y^2$.
8. $9a^2 + 6ab - 15ac - 10bc$.
9. $3x^3 - 9x^2y^2 + 5x^2 - 15xy^2 - 2x + 6y^2$.
10. $x^2y^2 - 4x^2 + 3xy^2 - 12x - 10y^2 + 40$ (four factors).
11. $10x^2 + 11xy - 14xz - 6y^2 + 17yz - 12z^2$.

OPERATION. This has the form of the product of two trinomials, each having terms in x, y, and z.

By Art. 95, the factors of the part $10x^2 + 11xy - 6y^2$ are $5x - 2y$ and $2x + 3y$.

The product of the terms not containing x and y is $-12z^2$. Hence $-12z^2$ must be resolved into two factors such that the sum of the products obtained by multiplying one of these factors by $5x$ and the other by $2x$ shall be $-14xz$; also the sum of the products obtained by multiplying one of these factors by $-2y$ and the other by $3y$ shall be $17yz$. By trial these are found to be $3z$ and $-4z$. Hence the factors sought are $5x - 2y + 3z$ and $2x + 3y - 4z$.

12. $x^2 + 3xy + 5x + 2y^2 + 8y + 6$.
13. $2x^2 - 5xy - 12y^2 - 9xz - 8yz + 4z^2$.
14. $6x^2 - 4xy - 12xz + 9xy - 6y^2 - 5yz + 6z^2$.
15. $12x^2 + 14xy - 11x - 10y^2 + 25y - 15$.
16. $6x^2 - 7xy - 3y^2 - 9x + 30y - 27$.
17. $4x^2 - 2xz - 9y^2 - 27yz - 20z^2$.
18. $15x^4 + x^2y^2 - 19x^2z - 6y^4 + 19y^2z - 10z^4$.

99. Theorem. *The difference of any two quantities is a divisor of the difference of the same powers of the quantities.*

The sum of two quantities is a divisor of the difference of the same even powers, and the sum of the same odd powers of the quantities.

DEM. 1st. Let x and y be any quantities, and n any positive integer. Then $x-y$ divides x^n-y^n.

Supplying the missing terms, changing the sign of the coefficient of the second term of the divisor, and proceeding with the division as in Art. 79, we have

$$\begin{array}{ccccc|l} x^n & +\,0\,x^{n-1}y & +\,0\,x^{n-2}y^2 & +\,0\,x^{n-3}y^3\cdots & -\,y^n & \underline{1} \\ & 1 & 1 & 1 & 0 & \end{array}$$

It is seen that the quantity to be added to the last term is 1, giving zero, and, consequently, that the division terminates, the quotient being

$$x^{n-1}+x^{n-2}y+x^{n-3}y^2+x^{n-4}y^3\cdots+y^{n-1}. \quad (1)$$

2d. Let n be even. Then $x+y$ divides x^n-y^n.

Proceeding as before, we have

$$\begin{array}{ccccc|l} x^n & +\,0\,x^{n-1}y & +\,0\,x^{n-2}y^2 & +\,0\,x^{n-3}y^3\cdots & -\,y^n & \underline{-1} \\ & -1 & 1 & -1 & & \end{array}$$

It is seen that if n is even, the quantity to be added to the last term is 1, giving zero, and, consequently, that the division terminates, the quotient being

$$x^{n-1}-x^{n-2}y+x^{n-3}y^2-x^{n-4}y^3\cdots-y^{n-1}. \quad (2)$$

3d. Let n be odd. Then $x+y$ divides x^n+y^n.

Proceeding as before, we have

$$\begin{array}{ccccc|l} x^n & +\,0\,x^{n-1}y & +\,0\,x^{n-2}y^2 & +\,0\,x^{n-3}y^3\cdots & +\,y^n & \underline{-1} \\ & -1 & 1 & -1 & & \end{array}$$

It is seen that if n is odd, the quantity to be added to the last term is -1, giving zero, and, consequently, that the division terminates, the quotient being

$$x^{n-1}-x^{n-2}y+x^{n-3}y^2-x^{n-4}y^3\cdots+y^{n-1},$$

which is the same as (2).

100. Sch. When the sum or difference of the same powers of two quantities is given, these theorems enable us to determine whether it can be factored, and, if so, what one of the factors is. Then the other can be written at once by the form (1) or (2).

EXAMPLES XXVI

Factor the following:

1. $x^5 + y^5$.

Operation. Since this is the sum of the same odd powers of two quantities, it is divisible by the sum of the quantities. Hence $x + y$ is one factor, and the other, as given in form (2) above, is $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.

2. $x^6 - y^6$.

Operation.
$$x^6 - y^6 = (x^3 + y^3)(x^3 - y^3)$$
$$= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2).$$

3. $x^6 + y^3$.

Operation.
$$x^6 + y^3 = (x^2)^3 + y^3 = (x^2 + y)[(x^2)^2 - (x^2)y + y^2]$$
$$= (x^2 + y)(x^4 - x^2y + y^2).$$

4. $243 - a^5$.

Operation.
$$243 - a^5 = 3^5 - a^5 = (3 - a)(3^4 + 3^3a + 3^2a^2 + 3a^3 + a^4)$$
$$= (3 - a)(81 + 27a + 9a^2 + 3a^3 + a^4).$$

5. $x^4 - y^4$.

6. $x^5 - y^5$.

7. $x^6 - 64$.

8. $27x^3 + y^3$.

9. $32x^5 - 1$.

10. $27x^3 + 125y^6$.

11. $8x^6 - 216$.

12. $a^9 + b^6$.

101. Prob. *To find by trial the binomial factors of a polynomial of the form* $Ax^n + Bx^{n-1} + Cx^{n-2} \cdots + L$, *or of a homogeneous polynomial of the form* $Ax^n + Bx^{n-1}y + Cx^{n-2}y^2 \cdots + Ly^n$.

Solution. The short method of successive division, given in Art. 81, affords a most useful method of factoring such expressions. Since the product of the factors of a polynomial will produce the polynomial, it is evident from the process of multiplication that the product of the last terms of the factors will

produce the last term of the polynomial. Hence in finding by trial these factors we need to try for the last terms only those numbers that are exact divisors of the last term of the polynomial. For example, let it be required to find the factors of

$$x^5 - 3x^4 - 5x^3 + 15x^2 + 4x - 12.$$

The binomial factors of the first degree are each $x \pm$ *some number* which is a factor of 12. A trial of the smaller factors of 12 results in the following:

$$\begin{array}{cccccc|l}
x^5 & -3x^4 & -5x^3 & +15x^2 & +4x & -12 & \underline{\ 1\ } \\
 & -2 & -7 & 8 & 12 & 0 & \underline{\ 2\ } \\
 & 0 & -7 & -6 & 0 & & \underline{\ 3\ } \\
 & 3 & 2 & 0 & & & \underline{-1} \\
 & 2 & 0 & & & & \underline{-2} \\
 & 0 & & & & &
\end{array}$$

It will be remembered that this is a short process of dividing successively by $x - 1$, $x - 2$, $x - 3$, $x + 1$, and $x + 2$. Hence, as each division is exact, the factors are $x - 1$, $x - 2$, $x - 3$, $x + 1$, and $x + 2$.

102. SCH. It will usually be found expedient to find the positive numbers first, as above. When all the coefficients of any row are plus, none but negative numbers need be tried, as the successive additions of positive products to positive numbers could not produce zero. Before writing down the various sums, it is best, if the coefficients are not too large, to run through mentally with a factor of the last term, and see whether the last addition gives zero.

103. NOTE. If the student has forgotten the short process of division as given in Art. 79, with its application in successive division as given in Art. 81, he should go back and thoroughly master it, as it is of frequent and important use in subsequent parts of the work.

EXAMPLES XXVII

Factor the following:

1. $x^5 - x^4 - 13x^3 + 13x^2 + 36x - 36.$

OPERATION

$$\begin{array}{cccccc|c}
x^5 - & x^4 - & 13x^3 + & 13x^2 + & 36x - & 36 & 1 \\
 & 0 & -13 & 0 & 36 & 0 & 2 \\
 & 2 & -9 & -18 & 0 & & 3 \\
 & 5 & 6 & 0 & & & -2 \\
 & 3 & 0 & & & & -3 \\
 & 0 & & & & &
\end{array}$$

The factors are $x-1$, $x-2$, $x-3$, $x+2$, and $x+3$.

2. $3x^4 + 13x^3 - 117x - 243.$

OPERATION

$$\begin{array}{ccccc|c}
3x^4 + & 13x^3 + & 0x^2 - & 117x - & 243 & 3 \\
 & 22 & 66 & 81 & 0 & -3 \\
 & 13 & 27 & 0 & &
\end{array}$$

The factors are $x-3$, $x+3$, and $3x^2+13x+27$.

3. $x^3 - 6x^2 + 11x - 6.$

4. $x^3 + 5x^2 + 3x - 9.$

5. $x^3 - 6x^2 + 13x - 10.$

6. $x^6 + 8x^2 + 17x + 10.$

7. $x^3 - 13x^2 + 49x - 45.$

8. $x^3 - 15x^2 + 74x - 120.$

9. $x^4 + 2x^3 - 3x^2 - 4x + 4.$

10. $x^4 - 10x^3 + 35x^2 - 50x + 24.$

11. $x^4 - 2x^3 - 25x^2 + 26x + 120.$

12. $x^4 - 4x^3 + 8x^2 - 8x - 21.$

13. $x^4 - 6x^3 + 5x^2 + 12x - 60.$

14. $x^5 - x^4 - 9x^3 + 5x^2 + 16x - 12.$

15. $x^5 - 9x^4 + 25x^3 - 15x^2 - 26x + 24.$

16. $x^5 - 4x^4 - 5x^3 + 20x^2 + 4x - 16.$

17. $2x^4 - 9x^3 + 4x^2 + 21x - 18.$

18. $5x^5 + 7x^4 - 21x^3 - 11x^2 + 32x - 12.$

19. $3x^5 - 2x^4 - 41x^3 + 56x^2 - 4x + 48.$

20. $x^6 - 14x^4 + 49x^2 - 36.$

21. $x^6 - 4x^5 - 3x^4 + 24x^3 - 10x^2 - 32x + 24.$

22. $x^7 + 5x^6 + 6x^5 - 6x^4 - 15x^3 - 3x^2 + 8x + 4.$

EXAMPLES XXVIII

Factor the following:

1. $7fg^2y - 28f^2gy^2 + 42f^3gy.$

2. $x^3y^3 - 7x^2y^4 + 12xy^5.$

3. $m^4 - n^4.$

4. $1 - 2\sqrt{x} + x.$

5. $256a^4 + 544a^2 + 289.$

6. $1 - c^3.$

7. $x^2 - x - 72.$

8. $y^6 - z^4.$

9. $x^3 + x^2 - 17x + 15.$

10. $x^4 - 9x^2 - 90.$

11. $\frac{25}{m^2} - \frac{40}{mx^2} + \frac{16}{x^4}.$

12. $\frac{a^2}{b^2} + \frac{b^2}{a^2} - 2.$

13. $125 + 64a^3.$

14. $x^3 - 15x^2 + 47x + 63.$

15. $a^2 + 23a + 22.$

16. $a^3 + b^3.$

17. $c^6 - d^6.$

18. $c^{-6} - d^{-6}.$

19. $1 - 13x^3 + 22x^6.$

20. $4x^2 + 8x + 3.$

21. $a^6 - b^{-6}.$

22. $x^2 + 6xy - 16y^2.$

23. $\frac{a^4}{x^4} - b^{-10}.$

24. $507m^4 + 1326m^2n^{\frac{3}{2}} + 867n^3.$

25. $\frac{9}{49}a^{4m} - \frac{2}{21}a^{2m}b^{2n+2} + \frac{1}{81}b^{4n+4}.$

26. $3a + 3b - 6\sqrt{ab}.$

27. $a^5 + b^5.$

28. $15a + 5ax - x - 3.$

29. $x^4 + 3x^3 - 15x^2 - 19x + 30.$

30. $12a^2x^{\frac{3}{2}} - 12a^2x^{\frac{3}{4}} + 3a^2.$

31. $x^4y^4 - 29x^2y^2 + 54.$

32. $21abcd - 28cdxy + 15abmn - 20mnxy.$

33. $2x^2 - 13x + 6.$

34. $3x^3 - 12x^3y^2 - 4y^2 + 1.$

35. $x^4 - 14x^3 + 32x^2 + 95x + 63.$

36. $x^2 - x - 9900.$

37. $x^2 + ax + x + a.$

38. $x^4 - 11x^2y + 18y^2.$

39. $6x^3 - 7ax^2 - 20a^2x.$

40. $x^5 - 20x^3 + 30x^2 + 19x - 30$

41. $x^{2m} + 31x^m - 32.$

42. $x^3 - x^2 - 2x - 2.$

43. $\frac{1}{x^{10}} - \frac{100}{y^{100}}.$

44. $10a\left[\frac{x^4}{y^2} + \frac{y^2}{x^4}\right] - 20a.$

45. $72\,cd^2m^3 - 84\,cd^3m^2 + 96\,c^2d^2m^2$. **46.** $10\,x^4 + 79\,x^2 - 8$.

47. $2\,x^7y + 54\,xy^4$. **48.** $6\,x^6 + 19\,x^3y^2 - 7\,y^4$.

49. $x^6 - x^5 - 20\,x^4 + 50\,x^3 - 11\,x^2 - 49\,x + 30$.

50. $9\,x^2 + 4\,y^2 + z^2 - 12\,xy + 6\,xz - 4\,yz$.

51. $3\,x^6 - 17\,x^5 + 23\,x^4 + 9\,x^3 - 14\,x^2 - 4\,x - 24$.

52. $9\,x^2 + 3\,xy - 2\,y^2 - 3\,x + 13\,y - 20$.

53. $x^2 + 12\,xy - 14\,xz + 36\,y^2 - 84\,yz + 49\,z^2$.

54. $2\,x^7 - 3\,x^6 - 14\,x^5 + 20\,x^4 + 28\,x^3 - 37\,x^2 - 16\,x + 20$.

55. $10\,x^2 + 16\,xy + 13\,xz - 8\,y^2 - 10\,yz - 3\,z^2$.

56. Is $x^{17} + y^{17}$ divisible by $x + y$? by $x - y$?

57. Is $x^{85} + y^{119}$ divisible by $x^5 - y^7$? by $x^5 + y^7$?

58. Write by form (2), Art. 99, the quotient of $a^{10} + b^{10}$ divided by $a^2 + b^2$.

CHAPTER VII

HIGHEST COMMON DIVISOR AND LOWEST COMMON MULTIPLE

SECTION I — HIGHEST COMMON DIVISOR

104. The Highest Common Divisor of two or more algebraical quantities is the quantity of highest degree that will exactly divide each of them.

The abbreviation **h. c. d.** or H. C. D. is often used for highest common divisor. The name *highest common factor,* with the abbreviation **h. c. f.** or H. C. F., is also used.

105. Quantities are said to be *prime to each other* when they have no common factor other than unity.

106. Note. As applied to literal quantities, highest common divisor is preferred to greatest common divisor, since, when numerical values are assigned, the quantity having the higher degree may be the smaller.

Thus, for positive values, if $a > 1$, $a^3 > a$; but if $a < 1$, $a^3 < a$. Again, for values of x and y greater than 1, if $x > y$, $x^2 - y^2 > x - y$; but if $x < y$, $x^2 - y^2 < x - y$.

107. Theorem. *The highest common divisor of two or more quantities is the product of their common prime factors.*

This follows directly from the definition.

EXAMPLES XXIX

Find the highest common divisor of each of the following:

1. $72\,a^4b^3c^2$, $84\,a^3b^2c$, and $180\,a^5b^4c^3$.

Solution.
$$\begin{aligned} 72\,a^4b^3c^2 &= 2^3 \cdot 3^2\,a^4b^3c^2, \\ 84\,a^3b^2c &= 2^2 \cdot 3 \cdot 7\,a^3b^2c, \\ 180\,a^5b^4c^3 &= 2^2 \cdot 3^2 \cdot 5\,a^5b^4c^3. \end{aligned}$$

The factors of highest degree common to all are 2^2, 3, a^3, b^2, and c. Hence the **h. c. d.** is the product of these, or $12\,a^3b^2c$.

2. $48\,a^2b^4$, $204\,a^3b^2$, and $228\,a^3b^3$.

3. $81\,x^3y^2z^4$, $123\,x^2y^4z^3$, and $315\,x^3y^3$.

4. $6\,m^2(x-y)^3$ and $9\,m^3(x-y)^5$.

5. $x^2y^3+2\,x^3y^2$ and $x^2y^4-4\,x^4y^2$.

6. $x^2+6\,x+8$ and $x^2+3\,x+2$.

7. x^2+x-6 and x^2-4.

8. x^2-1, $x^2-3\,x+2$, and $x^2+6\,x-7$.

9. x^3+1, x^2-1, and $x^2-2\,x-3$.

10. $2\,x^2-3\,x-2$ and $4\,x^2+8\,x+3$.

11. $a^2+2\,ab+b^2$, a^2-b^2, and a^3+b^3.

12. x^3-x, $x^3+9\,x^2-10\,x$, and x^6-x.

13. $x^3+3\,x^2y+2\,xy^2$ and $x^4+6\,x^3y+8\,x^2y^2$.

14. x^2-x-42, $x^2-4\,x-60$, and $x^2+12\,x+36$.

15. $2\,x^2-7\,x+3$ and $3\,x^2-7\,x-6$.

16. $x^4-2\,x^3-13\,x^2+38\,x-24$ and $x^4-4\,x^3-7\,x^2+34\,x-24$.

Operation. We proceed as follows, by the process of Art. 101 :

x^4	$-2\,x^3$	$-13\,x^2$	$+38\,x$	-24	1
	-1	-14	24	0	2
	1	-12	0		3
	4	0			-4
	0				

x^4	$-4\,x^3$	$-7\,x^2$	$+34\,x$	-24	1
	-3	-10	24	0	2
	-1	-12	0		4
	3	0			-3
	0				

The common factors are thus seen to be $x-1$ and $x-2$. Hence, multiplying by the process of Art. 60, the **h. c. d.** is $x^2-3\,x+2$.

17. $x^3+2\,x^2+x+2$ and $x^4-4\,x^2-x-2$.

18. $x^3+4\,x^2-8\,x+24$ and $x^4-x^3+8\,x-8$.

19. $2x^3 + x^2 - x - 2$ and $6x^3 - 4x^2 + 2x - 4$.

20. $x^4 - 5x^3 + 5x^2 - x - 12$ and $x^4 - 2x^3 - 12x^2 + 11x + 20$.

21. $x^3 - 13x + 12$ and $x^4 + 3x^3 + 12x - 16$.

22. $x^4 - 4x^3 - 7x^2 + 34x - 24$ and $x^5 - 6x^4 + x^3 + 36x^2 - 20x - 48$.

23. $x^6 - 1$ and $x^4 + x^3 - 9x^2 + 10x - 8$.

24. $12x^4 - 24x^3y + 12x^2y^2$ and $8x^3y^2 - 24x^2y^3 + 24xy^4 - 8y^5$.

25. $a^4 - a^3b - a^2b^2 - 2b^4$ and $3a^3 - 7a^2b + 3ab^2 - 2b^3$.

26. $a^4 - 5a^3b + 5a^2b^2 - ab^3 - 12b^4$ and $a^4 - 2a^3b - 12a^2b^2 + 11ab^3 + 20b^4$.

27. $3x^5 + 2x^4 - 47x^3 + 10x^2 + 128x - 96$ and $3x^5 - 10x^4 - 31x^3 + 94x^2 + 16x - 96$.

28. $x^5 - 5x^4 - 15x^3 + 65x^2 + 74x - 120$ and $x^5 - 4x^4 - 16x^3 + 46x^2 + 63x - 90$.

Operation. By the process of Art. 101, the common factors are found to be $x - 5$, $x + 3$, $x + 2$, and $x - 1$. After writing by Art. 60, the product of the first two, we proceed as follows, by the process of Art. 59:

$$\begin{array}{rrrrrr} (2) & 1 & -2 & -15 & & \\ (-1) & & 0 & -19 & -30 & \\ & & -1 & -19 & -11 & 30 \end{array}$$

Hence the **h. c. d.** is $x^4 - x^3 - 19x^2 - 11x + 30$.

29. $x^5 - 5x^4 + 7x^3 - 7x^2 + 16x - 12$ and $x^5 - 8x^4 + 26x^3 - 46x^2 + 45x - 18$.

30. $x^5 + 2x^4 - 15x^3 - 8x^2 + 68x - 48$ and $x^5 + 8x^4 + 15x^3 - 20x^2 - 76x - 48$.

31. $x^5 + 6x^4 + 6x^3 - 16x^2 - 15x + 18$ and $x^5 + 2x^4 - 10x^3 - 8x^2 + 33x - 18$.

32. $3x^5 + x^4 - 11x^3 + 3x^2 + 8x - 4$ and $3x^5 + 7x^4 - 3x^3 - 11x^2 + 4$.

33. $2x^6 + 5x^5 - 6x^4 - 4x^3 + 2x^2 - 29x + 30$ and $2x^5 - 7x^4 + 4x^3 - 4x^2 - 2x + 15$.

108. *When one of the polynomials is readily factored and the other not, we may find by trial what factors of the first are contained in the second, and thus obtain the* **h. c. d.** *of the two.*

EXAMPLES XXX

Find the **h. c. d.** of each of the following:

1. $x^3 + y^3$ and $5x^4 - 18x^3y + 12x^2y^2 - 7xy^3 - 6y^4$.

The factors of $x^3 + y^3$ (form (2) of Art. 99) are $x + y$ and $x^2 - xy + y^2$. By trial the second of these is found to be a factor of the second polynomial.

2. $x^3 - 1$ and $2x^3 - x^2 - x - 3$.

3. $9x^2 - 16$ and $9x^3 - 15x^2 + 10x - 8$.

4. $x^3 + 2x^2y + 4xy^2 + 3y^3$ and $x^4 + x^3y + 4x^2y^2 + xy^3 + 3y^4$.

5. $x^5 + y^5$ and $7x^5 - 10x^4y + 10x^3y^2 - 10x^2y^3 + 10xy^4 - 3y^5$.

109. Note. When the polynomials, or one of them, can be readily factored, the processes of Arts. 107 and 108 of finding the **h. c. d.** are the most expeditious. Otherwise, the process of Art. 111, similar to that used for numbers in Arithmetic, may be employed.

110. Theorem. *1st. A divisor of a quantity is a divisor of any multiple of that quantity.*

2d. A common divisor of two quantities is a divisor of their sum and also of their difference.

The first is self-evident.

For the second, let d be a common divisor of a and b, and q and q' the respective quotients. Then

$$a = qd,$$

and $$b = q'd,$$

whence $$a \pm b = qd \pm q'd = (q \pm q')d.$$

Hence d is a divisor of $a \pm b$.

111. Prob. *To find the* **h. c. d.** *of two polynomials which are not readily resolved into their prime factors.*

RULE. *Having arranged the polynomials with reference to the same letter, remove, and reserve as factors of the* **h. c. d.**, *any monomial factors common to both polynomials, and reject from each polynomial all other monomial factors.*

Divide the reduced polynomial of higher degree in the letter of arrangement by the other (if both are of the same degree, either may be used as the dividend), first multiplying the dividend, if necessary, by any number that will avoid a fraction in the quotient.

Reject from the remainder any monomial factors and divide the former divisor by this reduced remainder, first multiplying, if necessary, by any number that will avoid a fraction in the quotient.

Continue the process either until there is no remainder, or until the letter of arrangement disappears from the remainder. In the latter case the two reduced polynomials are prime to each other, and in the former case the product of the last divisor and the reserved common monomial factors will be the **h. c. d.** *of the given polynomials.*

DEM. Rejecting from or introducing into either polynomial or any remainder a monomial factor will not affect the **h. c. d.**, since the **h. c. d.** is composed of those factors only that are common to the two polynomials.

If A and B represent the reduced polynomials, q, q', etc., the successive quotients, and R, R', etc., the successive remainders, and if we suppose the third remainder to be 0, the work (except rejecting and introducing factors) will stand as in the margin.

$$
\begin{array}{l}
B)A(q \\
\underline{qB} \\
\quad R)B(q' \\
\quad \underline{q'R} \\
\qquad R')R(q'' \\
\qquad \underline{q''R'} \\
\qquad\quad 0
\end{array}
$$

A divisor of A and B is (Art. 110) a divisor of $A - qB$, or R; and a divisor of B and R is a divisor of $B - q'R$, or R'. Therefore any divisor of A and B is a divisor of R'; and since R' is its own highest divisor, it is the highest common divisor of A and B. Hence the product of R' (the last divisor), and the reserved common monomial factors is the **h. c. d.** of the given polynomials.

EXAMPLES XXXI

Find the **h. c. d.** of the following:

1. $8x^5 - 48x^4y + 84x^3y^2 - 40x^2y^3$ and

$$6x^5y - 48x^4y^2 + 126x^3y^3 - 120x^2y^4 + 24xy^5.$$

Operation

Removing $4x^2$ from the first and $6xy$ from the second, reserving $2x$, the **h. c. d.** of these, as a part of the **h. c. d.** of the polynomials, we have left

$$2x^3 - 12x^2y + 21xy^2 - 10y^3 \text{ and } x^4 - 8x^3y + 21x^2y^2 - 20xy^3 + 4y^2.$$

Multiplying the second by 2 to avoid fractions, and using Synthetic Division (in which the signs of all the terms of the divisor except the first must be changed), we have

$$\begin{array}{ccccc|l}
2x^4 - & 16x^3y + & 42x^2y^2 - & 40xy^3 + & 8y^4 & \underline{2x^3 + 12x^2y - 21xy^2 + 10y^3} \\
 & 12 & -21 & 10 & -20 & x - 2y \\
 & -4 & -24 & 42 & & \\
 & & -3 & 12 & -12 &
\end{array}$$

$\therefore$ the remainder is $-3x^2 + 12xy - 12y^2$.

Rejecting -3, we have for the new divisor $x^2 - 4xy + 4y^2$.

$$\begin{array}{cccc|l}
2x^3 - & 12x^2y + & 21xy^2 - & 10y^3 & \underline{x^2 + 4xy - 4y^2} \\
 & 8 & -8 & 16 & 2x - 4y \\
 & -4 & -16 & & \\
 & & -3 & 6 &
\end{array}$$

$\therefore$ the remainder is $-3x + 6y$.

Rejecting -3, we have for the next divisor $x - 2y$.

$$\begin{array}{ccc|l}
x^2 - & 4xy + & 4y^2 & \underline{2} \\
 & -2 & 0 &
\end{array}$$

Hence, in dividing by $x - 2y$ there is no remainder. Multiplying this last divisor by the common factor $2x$ reserved at the beginning, we have for the **h. c. d.** of the original polynomials $2x(x - 2y) = 2x^2 - 4xy$.

2. $3x^3 + 9x^2y - 6xy^2 - 6y^3$ and $24x^3 + 6x^2y - 12xy^2 - 18y^3$.

3. $21x^3 - 32x^2 - 54x - 7$ and $21x^3 - 4x^2 - 15x - 2$.

4. $10x^3 + x^2 - 9x + 24$ and $20x^4 - 17x^2 + 48x - 3$.

5. $2x^4 - 7x^3 + 11x^2 - 8x + 2$ and $2x^4 + x^3 - 9x^2 + 8x - 2$.

6. $4x^5 + 14x^4 + 20x^3 + 70x^2$ and $8x^7 + 28x^6 - 8x^5 - 12x^4 + 56x^3$.

112. **Prob.** *To find the* **h. c. d.** *of three or more polynomials.*

Rule. *Find the* **h. c. d.** *of any two of the given polynomials, then find the* **h. c. d.** *of this* **h. c. d.** *and a third polynomial, and so on until all have been used.*

Dem. If A, B, C, etc., represent the given polynomials, and M the **h. c. d.** of A and B, M contains all the factors common to A and B (Art. 107). If N represent the **h. c. d.** of M and C, N contains all the factors common to M and C, and consequently all the factors common to A, B, and C; and so on.

EXAMPLES XXXII

Find the **h. c. d.** of the following:

1. $x^2 + 11x + 30$, $2x^2 + 21x + 54$, and $9x^3 + 53x^2 - 9x - 18$.

2. $x^4 - 7x^3 + 5x^2 + 31x - 30$, $x^4 - 9x^3 + 21x^2 + x - 30$, and $x^4 - 8x^3 + 6x^2 + 72x - 135$.

3. $10x^5 + 10x^3y^2 + 20x^4y$, $2x^3 + 2y^3$, and $4y^4 + 12x^2y^2 + 4x^3y + 12xy^3$.

SECTION II—LOWEST COMMON MULTIPLE

113. **The Lowest Common Multiple** of two or more algebraical quantities is the quantity of lowest degree that is exactly divisible by each of them.

The abbreviation **l. c. m.** or L. C. M. is often used for lowest common multiple.

114. **Prob.** *To find the* **l. c. m.** *of two or more algebraical quantities.*

Rule. *Multiply one of the quantities (the one of highest degree when they differ in degree) by all the factors of the others that are not found in it.*

Dem. Let the quantities be represented by A, B, C, etc. As any one of the quantities, A for example, is its own lowest multiple, the **l. c. m.** of all the quantities must contain it as a factor. As the **l. c. m.** contains also B, C, etc., it must contain all the factors of these quantities; hence any of these factors that are not found in A must be introduced.

115. Sch. In applying this rule, if the factors of the polynomials are not readily found, we may find the **h. c. d.** in the usual way, and then find the other factors by dividing each polynomial by the **h. c. d.**

EXAMPLES XXXIII

Find the **l. c. m.** of the following:

1. x^3-2x^2-5x+6, x^3-3x^2-x+3, and x^3+4x^2+x-6.

Solution. Factoring by the method of Art. 101, we have

$$x^3-2x^2-5x+6=(x-1)(x-3)(x+2),$$
$$x^3-3x^2-x+3=(x-1)(x-3)(x+1),$$
$$x^3+4x^2+x-6=(x-1)(x+2)(x+3).$$

The second and third polynomials contain one factor each not found in the first. Hence multiplying the first polynomial by these factors, using the method of Art. 59, we have for the **l. c. m.**

$$(x^3-2x^2-5x+6)(x+1)(x+3)=x^5+2x^4-10x^3-20x^2+9x+18.$$

2. x^2+2, x^2-2, and x^4-4.

3. x^2-y^2, $(x+y)^2$, $(x-y)^2$, and $(x+y)^3$.

4. x^2-9 and $x^2+10x+21$.

5. $x^2-2x-15$ and $x^2-4x-21$.

6. $x^2-3x-70$ and $x^3-39x+70$.

7. x^2-9, x^2+x-12, and $x^2+2x-15$.

8. x^2+x-2, x^2-x-6, and x^2-4x+3.

9. $2x^2-7x-4$, $4x^2+10x+4$, and $6x^2-7x-5$.

10. a^3-2a^2-5a+6, a^3-3a^2-a+3, and a^3+4a^2+a-6.

11. $x^3-2x^2y+4xy^2-8y^3$, $x^3+2x^2y+4xy^2+8y^3$, and x^2-4y^2.

12. $x^4+5x^3+5x^2-5x-6$, $x^3+6x^2+11x+6$, and x^3+4x^2+x-6.

13. a^4-1, a^3+a^2+a+1, a^3-a^2+a-1, and a^2+1.

14. x^4-10x^2+9, $x^4+10x^3+20x^2-10x-21$, and $x^4+4x^3-22x^2-4x+21$.

15. $6x^2-5xy+2x-6y^2+23y-20$ and $8x^2-14xy+22x+3y^2-13y+12$.

CHAPTER VIII

FRACTIONS

116. **An Algebraic Fraction** is an indicated operation in division when written in the form $\frac{m}{n}$ or m/n.

In this form the dividend, written above or before the vinculum, is called the **Numerator**, the divisor is called the **Denominator**, and the quotient is called the **Value of the Fraction.**

117. When the denominator is a positive integer, it indicates, as in case of arithmetical fractions, the number of equal parts into which a unit is conceived to be divided, and the numerator indicates the number of these parts taken; but it would be absurd to represent a unit as divided into, say $5\frac{3}{4}$ equal parts, or into -4 equal parts. Now in such a fraction as $\frac{m}{n}$, m and n are unrestricted in value, and are not necessarily positive integers. Hence the denominator of an algebraic fraction does not necessarily indicate into how many equal parts a unit is conceived to be divided.

118. A quantity is said to have the **Integral Form** when it has no part in the fractional form.

Thus, $3\,mn$, $2\,x-3\,y$, $ax^3-bx^2y+cy^3$, are in the integral form.

119. A quantity is said to have the **Mixed Form** when it contains terms in both the integral and the fractional form.

Thus, $a+\frac{m}{n}$, $x^2-y+\frac{x-y^3}{x^2-y}$, have the mixed form.

120. **A Proper Fraction** is one which cannot, without the use of negative exponents, be reduced to the integral or mixed form. If its numerator and denominator contain a common letter, the numerator is of lower degree in this letter than the denominator.

Thus, $\dfrac{x+y}{z}$, $\dfrac{x^2+5}{x^3+3x^2-7}$, are proper fractions.

121. **An Improper Fraction** is one whose numerator is not of lower degree in a common letter than its denominator.

Thus, $\dfrac{x+5}{x+6}\left(=1-\dfrac{1}{x+6}\right)$, $\dfrac{x^3+3x^2-4x+5}{x^2-2x+3}\left(=x+5+\dfrac{3x-10}{x^2-2x+3}\right)$, are improper fractions.

122. **A Simple Fraction** is a fraction whose numerator and denominator are both in the integral form.

123. **A Complex Fraction** is a fraction having its numerator or its denominator or both in the fractional or mixed form.

124. **Reduction** is the operation of changing the form of a quantity without changing its value.

125. **The Lowest Common Denominator** of several fractions is the lowest common multiple (Art. 113) of their denominators.

126. A fraction is in its **Lowest Terms** when its numerator and denominator are prime to each other.

REDUCTION OF FRACTIONS

127. **Theorem.** *Multiplying or dividing both numerator and denominator of a fraction by the same quantity does not change the value of the fraction.*

Dem. Represent numerator, denominator, and quotient by n, d, and q, respectively.

Since divisor times quotient equals dividend (Art. 66), we have

$$d \times q = n.$$

Since multiplying or dividing a factor of a product multiplies or divides the product, multiplying or dividing d and n in the above equation by the same quantity will not change q.

128. **Cor.** *Changing the signs of all the terms in a fraction does not change the value of the fraction.*

For this is equivalent to multiplying or dividing both numerator and denominator by -1.

129. Prob. *To reduce a fraction to its lowest terms.*

RULE. *Reject from numerator and denominator all factors common to both; or,* what is the same thing, *reject from both their* **h. c. d.**

This follows from Arts. 127 and 107.

130. NOTE. The student should endeavor to find the factors of numerator and denominator, and after finding them he should cancel those that are common, avoiding whenever possible the laborious method of Art. 111 of finding the **h. c. d.** In none of the following is it necessary to resort to this method. In the last seven he should factor by the method of Art. 101.

EXAMPLES XXXIV

Reduce the following to their lowest terms:

1. $\dfrac{60\,ab^3c^2}{84\,ab^2c^3}$.

2. $\dfrac{420\,a^2bc^{2n}}{630\,ab^2c^n}$.

3. $\dfrac{210\,a^{\frac{1}{2}}b^{\frac{3}{2}}c^{2n}}{330\,a^{\frac{1}{3}}b^{\frac{1}{2}}c^{5n}}$.

4. $\dfrac{bx + x^2}{ab + ax}$.

5. $\dfrac{x^2 - y^2}{x^4 - y^4}$.

6. $\dfrac{3\,a^3b - 6\,a^2b^2}{4\,a^2b^2 - 8\,ab^3}$.

7. $\dfrac{x^2 + 7\,x + 10}{x^2 + 4\,x - 5}$.

8. $\dfrac{x^2 - x - 12}{x^2 + x - 20}$.

9. $\dfrac{4\,x^2 - 4\,x + 1}{4\,x^3 - 3\,x + 1}$.

10. $\dfrac{x^2 + 5\,x - 14}{x^2 + 10\,x + 21}$.

11. $\dfrac{(1 + x)^3}{(1 - x^2)^2}$.

12. $\dfrac{a^3 + x^3}{a^2 + 2\,ax + x^2}$.

13. $\dfrac{x^3 - 27}{x^2 - 2\,x - 3}$.

14. $\dfrac{x^2 - 7\,x + 10}{x^2 - 5\,x + 6}$.

15. $\dfrac{x^2 - x^2y^2}{(x + xy)^2}$.

16. $\dfrac{1 - 5\,a + 6\,a^2}{1 - 7\,a + 12\,a^2}$.

17. $\dfrac{x^4y - xy^4}{x^3y^2 + x^2y^3 + xy^4}$.

18. $\dfrac{(a + b)^2 - c^2}{(a + b + c)^2}$.

19. $\dfrac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2}$. **20.** $\dfrac{6x^2-5x-4}{6x^2+x-12}$.

21. $\dfrac{(b+c+d)^2-a^2}{(a-b)^2-(c+d)^2}$. **22.** $\dfrac{x^3-7x+6}{x^3-6x^2+11x-6}$.

SUG. Factor by the process of Art. 101.

23. $\dfrac{x^3-8x^2+19x-12}{x^3-10x^2+29x-20}$.

24. $\dfrac{2x^4-x^3-9x^2+13x-5}{7x^4-26x^3+36x^2-22x+5}$.

25. $\dfrac{x^3-8x^2+21x-18}{3x^4-22x^3+53x^2-42x}$.

26. $\dfrac{x^4+10x^3+35x^2+50x+24}{x^4+x^3-19x^2-49x-30}$.

27. $\dfrac{2x^5-9x^4+8x^3+15x^2-28x+12}{3x^5-19x^4+45x^3-49x^2+24x-4}$.

28. $\dfrac{6x^5-19x^4+6x^3+36x^2-44x+15}{8x^5-18x^4-3x^3+37x^2-33x+9}$.

131. Prob. *To reduce an improper fraction to an integral or mixed form.*

RULE. *Divide the numerator by the denominator, continuing the division either until it terminates, or until the remainder is of lower degree than the denominator.*

EXAMPLES XXXV

Reduce the following to integral or mixed forms:

1. $\dfrac{6x^3-4x^2-18x-10}{2x^2-4x-1}$.

OPERATION. Dividing by the method of Art. 76, we have

$$\begin{array}{cccc|c}
6x^3 & -\,4x^2 & -\,18x & -\,10 & 2x^2+4x+1 \\ \hline
 & \underline{12} & 3 & 4 & 3x+4 \\
 & 8 & \underline{16} & & \\
 & & 1 & \overline{-6} &
\end{array}$$

Hence the result is $3x+4+\dfrac{x-6}{2x^2-4x-1}$.

2. $\dfrac{6x^2-12x-2}{3x}$.

3. $\dfrac{3x^3-8x^2+6x+3}{x-2}$.

SUG. In such examples as the 3d and 4th employ the method of Art. 101.

4. $\dfrac{2x^3-4x^2y+4xy^2-25y^3}{x-3y}$.

5. $\dfrac{m^2+n^2+2mn-x-y}{m+n}$.

6. $\dfrac{12x^3-8x^2+4x-5}{4x^2+3}$.

7. $\dfrac{32x^5+243}{2x+3}$.

8. $\dfrac{2x^3-3x^2-5}{x^2-x-1}$.

9. $\dfrac{8x^4+16x^3-10x^2-28x+11}{2x^2+x-3}$.

10. $\dfrac{10x^4+7x^3-11x^2+13x-3}{2x^3+7x^2+5x-2}$.

11. $\dfrac{x^4+4x^3y+6x^2y^2+4xy^3+y^4}{x^3+3x^2y+3xy^2+y^3}$.

132. Prob. *To reduce a mixed quantity to the fractional form.*

RULE. *Multiply the integral part by the denominator of the fractional part, to the product add the numerator of the fractional part, and place the sum over the denominator of the fractional part.*

DEM. The value of the integral part is not changed by multiplying it by the denominator of the fractional part and then indicating its division by the same quantity. The two numerators may now be added and written over the common denominator, since the quotient of the sum is equal to the sum of the quotients, the divisor being the same (Art. 72).

133. Cor. *An integer may be reduced to the form of a fraction with any denominator by multiplying it by that denominator and then indicating its division by the same quantity.*

EXAMPLES XXXVI

Reduce the following to the fractional form:

1. $ax^2+\dfrac{b^3-a^2x^2}{a}$.

2. $3x+4+\dfrac{9x^2+16}{3x-4}$.

3. $a-x-\dfrac{(a+x)^2}{a-x}$.

4. $x^2-xy+y^2-\dfrac{2x^3}{x+y}$.

5. $a^3 + a^2b + ab^2 + b^3 + \dfrac{2\,b^4}{a-b}$.

6. $1 + \dfrac{b^2 + c^2 - a^2}{2\,bc}$.

7. $x - 2 + \dfrac{11\,x + 22}{x^2 + 7\,x + 10}$.

8. $a - b + \dfrac{3\,a^2b + 2\,ab^2 - b^3}{a^2 - b^2}$.

134. Prob. *To reduce fractions having different denominators to equivalent fractions having the lowest common denominator.*

Rule. *Divide the* **l. c. m.** *of all the denominators by the denominator of each fraction, and multiply its numerator by the quotient.*

Dem. The object of the division is merely to find the factor by which the denominator of the fraction must be multiplied to produce the lowest common denominator. As the numerator is multiplied by the same factor, the value of the fraction is not changed.

135. Cor. *When the denominators have no common factors, the lowest common denominator is the product of all the denominators, and each numerator is multiplied by the denominators of all the other fractions.*

EXAMPLES XXXVII

Reduce the following to equivalent fractions having the lowest common denominator:

1. $\dfrac{x}{a+b},\ \dfrac{y}{a-b},\ \dfrac{z}{a^2-b^2}$.

2. $\dfrac{5}{2-2\,x^2},\ \dfrac{1}{1-x^2},\ \dfrac{2}{3\,x-3}$.

3. $\dfrac{x}{x+y},\ \dfrac{x^2}{x^2-y^2},\ \dfrac{x^3}{x^4-y^4}$.

4. $\dfrac{x}{a^2-b^2},\ \dfrac{y}{(a+b)^2},\ \dfrac{z}{(a-b)^2}$.

5. $\dfrac{2\,x}{x^2-xy+y^2},\ \dfrac{3\,y}{x^3+y^3},\ \dfrac{4\,z}{x+y}$.

6. $\dfrac{1}{x-y},\ \dfrac{x}{(x-y)^2},\ \dfrac{x^2}{(x-y)^3}$.

7. $\dfrac{1}{x^2+x-6},\ \dfrac{2}{x^2+5\,x+6},\ \dfrac{3}{x^2-4}$.

8. $\dfrac{1}{x+2},\ \dfrac{1}{x^2+4\,x+4},\ \dfrac{1}{x^3+6\,x^2+12\,x+4}$.

9. $\dfrac{(x-2)(x+1)}{x^2+x-2},\ \dfrac{(x+2)(x-1)}{x^2-x-2},\ \dfrac{(x+2)(x+1)}{x^2-3\,x+2},\ \dfrac{(x-2)(x-1)}{x^2+3\,x+2}$.

ADDITION AND SUBTRACTION OF FRACTIONS

136. Prob. *To add or subtract fractions.*

Rule. *Reduce the fractions to equivalent fractions having the lowest common denominator, write the sum or difference of the numerators over the lowest common denominator, and reduce the result to its simplest form.*

Dem. After the first step, which does not change the values of the fractions (Art. 134), we apply the principle that the sum or difference of the quotients is equal to the quotient of the sum or difference, the divisors being the same (Art. 72).

EXAMPLES XXXVIII

Perform the operations indicated in the following:

1. $\dfrac{a^2+x^2}{x+a^2}+\dfrac{a^2x}{x-a^2}-\dfrac{x^3+a^4x}{x^2-a^4}.$

Operation. Reducing to equivalent fractions having a common denominator, we have

$$\frac{x^3-a^2x^2+a^2x-a^4}{x^2-a^4}+\frac{a^2x^2+a^4x}{x^2-a^4}-\frac{x^3+a^4x}{x^2-a^4}=\frac{a^2x-a^4}{x^2-a^4}=\frac{a^2(x-a^2)}{x^2-a^4}=\frac{a^2}{x+a^2}.$$

2. $\dfrac{1}{x+y}+\dfrac{1}{x-y}.$

3. $\dfrac{1}{x+x^2}+\dfrac{1}{x-x^2}.$

4. $\dfrac{a^2+ab+b^2}{a+b}-\dfrac{a^2-ab+b^2}{a-b}.$

5. $\dfrac{2x+x^2-x^3}{4x^2-4x^3+x^4}-\dfrac{x}{2x-x^2}.$

6. $\dfrac{2}{x^3+x^2+x+1}+\dfrac{3}{x^3-x^2+x-1}.$

7. $\dfrac{x-a}{x+a}+\dfrac{a^2+3ax}{a^2-x^2}+\dfrac{x+a}{x-a}.$

8. $\dfrac{1}{x-3}+\dfrac{1}{x^2-5x+6}-\dfrac{2}{x^2-6x+8}.$

9. $\dfrac{1}{x+3}+\dfrac{7}{x^2-x-12}+\dfrac{x+3}{x-4}.$

10. $\dfrac{3x^2-8}{x^3-1}-\dfrac{5x+7}{x^2+x+1}+\dfrac{2}{x-1}.$

11. $\dfrac{x-3}{x^2-3x+2}-\dfrac{x-2}{x^2-4x+3}-\dfrac{x-1}{x^2-5x+6}.$

12. $\dfrac{10x^2}{(1+x^2)(1-4x^2)}+\dfrac{2}{1+x^2}-\dfrac{1}{1-2x}.$

13. $\dfrac{x+2}{x^3-x}-\dfrac{1}{2x+2}-\dfrac{3}{2x-2}+\dfrac{2}{x}.$

14. $\dfrac{x^2-1}{x^4+x^2+1}-\dfrac{x+1}{x^2-x+1}+\dfrac{x-1}{x^2+x+1}.$

15. $\dfrac{1}{x^2-1}+\dfrac{1}{(x+1)^2}+\dfrac{x(x^2+3)}{(x^2-1)^2}-\dfrac{2}{(x-1)^2}.$

16. $\dfrac{1}{x+3}+\dfrac{x+1}{x^2-3x+9}+\dfrac{x^2+x+1}{x^3+27}.$

17. $\dfrac{x^2-2x-15}{x^3-5x^2-9x+45}+\dfrac{x^2+7x+12}{x^3+3x^2-9x-27}-\dfrac{x^2-16}{x^3-4x^2-9x+36}.$

MULTIPLICATION OF FRACTIONS

137. Theorem. *1st. Multiplying the numerator of a fraction multiplies the fraction.*

2d. Dividing the numerator of a fraction divides the fraction.

3d. Multiplying the denominator of a fraction divides the fraction.

4th. Dividing the denominator of a fraction multiplies the fraction.

Dem. The numerator is dividend, or quantity to be measured, and the denominator is divisor, or measure. Now multiplying the quantity to be measured multiplies the number of times it contains the measure, and dividing the quantity to be measured divides the number of times it contains the measure.

Likewise, multiplying the measure divides the number of times it is contained in the quantity to be measured, and dividing the measure multiplies the number of times it is contained in the quantity to be measured.

138. Prob. *To multiply a fraction by an integer.*

Rule. *1st. Cancel all factors common to the integer and the denominator.*

2d. Multiply the numerator by the remaining factors of the integer.

This follows from Arts. 127 and 137.

139. Prob. *To multiply a fraction by a fraction.*

RULE. *1st. Cancel all factors common to either numerator and either denominator.*

2d. Take the product of the remaining factors of the numerators for the numerator of the result, and the product of the remaining factors of the denominators for the denominator of the result.

DEM. Let it be required to multiply $\frac{a}{b}$ by $\frac{m}{n}$, these fractions being supposed to be simple, since if the given fractions are not simple, they may be made so without changing their values.

Multiplying by m gives (Art. 137) $\frac{am}{b}$. But we were to multiply, not by m, but by one nth of m; hence this product must be divided by n, which gives (Art. 137) $\frac{am}{bn}$.

The rejection of all factors common to the numerator and denominator will evidently leave the result the same as it would have been had they been rejected before the multiplication. Their rejection before multiplying leaves simpler quantities to multiply.

EXAMPLES XXXIX

Multiply the following:

1. $\dfrac{x^2+2x-3}{x^2+5x+6}$ by $\dfrac{4x^2-12x-40}{3x^2-18x+15}$.

OPERATION.
$$\frac{x^2+2x-3}{x^2+5x+6}\times\frac{4x^2-12x-40}{3x^2-18x+15}$$
$$=\frac{(x+3)(x-1)}{(x+3)(x+2)}\times\frac{4(x-5)(x+2)}{3(x-5)(x-1)}=\frac{4}{3}.$$

2. $\dfrac{3a^2b^5}{28x^4y^2}$ by $21xy^3$.

3. $\dfrac{a}{x^2-y^2}$ by $x-y$.

4. $\dfrac{x+4}{x^2+4x-21}$ by $x+7$.

5. $\dfrac{3}{x^3-7x+6}$ by x^2+2x-3.

SUG. Factor the denominator of the 5th by the method of Art. 101, first supplying the missing term, $0x^2$.

6. $\dfrac{3\,a^2b^3}{4\,c^4} \times \dfrac{6\,b^2c^3}{5\,a^6} \times \dfrac{10\,ac^2}{9\,b^3}.$

7. $\dfrac{a}{a^4 - b^6}$ by $\dfrac{a^2 + b^3}{a^2 - b^3}.$

8. $\dfrac{9\,x^2 - 1}{x^3 - 25\,x}$ by $\dfrac{x^2 + 5\,x}{3\,x - 1}.$

9. $\dfrac{a^2 - 1}{(a + 1)^2}$ by $\dfrac{a^6 - 1}{(a^2 - a)^2}.$

10. $\dfrac{x^2 + 3\,x - 18}{x^2 - 8\,x + 12}$ by $\dfrac{2\,x^3 - 4\,x^2}{x^2 - 6}.$

11. $\dfrac{5\,x + 15}{8\,x - 4} \times \dfrac{3\,x - 9}{10\,x + 5} \times \dfrac{8\,x^2 - 2}{3\,x^2 - 27}.$

12. $\dfrac{1 - x^2}{1 + y} \times \dfrac{1 - y^2}{x + x^2} \times \left(1 + \dfrac{x}{1 - x}\right).$

13. $\dfrac{x^2 - x - 6}{x^2 + 4\,x + 4}$ by $\dfrac{x^2 - 2\,x - 8}{x^2 - 7\,x + 12}.$

14. $\dfrac{x^2 + y^2}{x^2 - xy}$ by $\dfrac{xy - y^2}{x^4 - y^4}.$

15. $\dfrac{x}{2}\left(\dfrac{1}{x - y} - \dfrac{1}{x + y}\right)$ by $\dfrac{x^2 - y^2}{x^2y + xy^2}.$

16. $\dfrac{x^2 + xy - 2\,y^2}{x^2 - 5\,xy + 4\,y^2}$ by $\dfrac{x^2 - 7\,xy + 12\,y^2}{x^2 + 5\,xy + 6\,y^2}.$

17. $\dfrac{x^2}{x^2 - y^2} - \dfrac{y^2}{x^2 + y^2}$ by $\dfrac{(x^2 - y^2)^2}{(x^2 - y^2)^2 + (x^2 + y^2)^2}.$

18. $\dfrac{x^3 + 4\,x^2 + 4\,x + 3}{x^3 - 4\,x^2 + 2\,x + 1}$ by $\dfrac{x^3 - 5\,x^2 + 5\,x + 2}{x^3 - x^2 - x - 2}.$

19. $\dfrac{3\,x^4 - 17\,x^3 + 27\,x^2 - 7\,x - 6}{2\,x^4 + 13\,x^3 + 28\,x^2 + 23\,x + 6}$ by $\dfrac{2\,x^3 + 7\,x^2 + 7\,x + 2}{x^3 - 6\,x^2 + 11\,x - 6}.$

20. $\dfrac{x^5 - 3\,x^4 - 5\,x^3 + 15\,x^2 + 4\,x - 12}{x^4 - x^3 - 19\,x^2 + 49\,x - 30}$ by

$$\frac{x^4 - 2\,x^3 - 13\,x^2 + 38\,x - 24}{x^5 - 2\,x^4 - 10\,x^3 + 20\,x^2 + 9\,x - 18}$$

21. $\dfrac{10\,x^5 - 15\,x^3 - 14\,x^2 + 21}{15\,x^7 - 21\,x^4 + 25\,x^3 - 35}$ by $\dfrac{3\,x^5 - 9\,x^4 + 5\,x - 15}{2\,x^3 - 6\,x^2 - 3\,x + 9}.$

DIVISION OF FRACTIONS

140. Prob. *To divide a fraction by an integer.*

RULE. *1st. Cancel all factors common to the integer and the numerator.*

2d. Multiply the denominator by the remaining factors of the integer.

This follows from Arts. 127 and 137.

141. Prob. *To divide a fraction by a fraction.*

RULE. *Invert the divisor and proceed as in multiplication.*

DEM. Let it be required to divide $\frac{a}{b}$ by $\frac{m}{n}$, these fractions being supposed to be simple, since if the given fractions are not simple, they may be made so without changing their values.

Dividing by m gives (Art. 137) $\frac{a}{bm}$. But we were to divide, not by m, but by one nth of m; hence, since the divisor used is n times what it should be, the quotient is one nth of what it should be, and must be multiplied by n, which gives (Art. 137) $\frac{an}{bm}$. We observe that in the operation we have inverted the divisor and proceeded as in multiplying a fraction by a fraction.

EXAMPLES XL

Divide the following:

1. $\frac{35\,a^3b^4}{6\,xy^2}$ by $14\,a^2b^2$.
2. $\frac{x^4-9\,y^2}{x+4\,y}$ by $x^2+3\,y$.
3. $\frac{x^3-y^3}{3\,ax}$ by x^2+xy+y^2.
4. $\frac{a^{2n}-b^{2m}}{ab}$ by a^n-b^m.
5. $\frac{a^2-b^2}{a+2b}$ by $\frac{a-b}{3\,a+6\,b}$.
6. $\frac{x^2+10\,x+21}{x^3-4\,x^2+3\,x}$ by $\frac{x^2-9}{x^3-x^2}$.
7. $\frac{x^3-y^3}{x^3+y^3}$ by $\frac{x-y}{x+y}$.
8. $\frac{x}{a}-\frac{a}{x}$ by $\frac{(a+x)^2}{ax}$.
9. $\frac{x^2+4\,xy+4\,y^2}{x-y}$ by $\frac{xy+2\,y^2}{x^2-xy}$.
10. $\frac{4\,(a^2-ab)}{b\,(a+b)^2}$ by $\frac{6\,ab}{a^2-b^2}$.

11. $a^2 - b^2 - c^2 - 2\,bc$ by $\dfrac{a+b+c}{a+b-c}$.

12. $\dfrac{2\,x^2 + 13\,x + 15}{4\,x^2 - 9}$ by $\dfrac{2\,x^2 + 11\,x + 5}{4\,x^2 - 1}$.

13. $\dfrac{x^2 - 14\,x - 15}{x^2 - 4\,x - 45}$ by $\dfrac{x^2 - 12\,x - 45}{x^2 - 6\,x - 27}$.

14. $5 - \dfrac{a^2 - 19\,x^2}{a^2 - 4\,x^2}$ by $3 - \dfrac{a - 5\,x}{a - 2\,x}$.

15. $\dfrac{x + 2\,y}{x + y} + \dfrac{x}{y}$ by $\dfrac{x + 2\,y}{y} - \dfrac{x}{x + y}$.

16. $\dfrac{x^2 - 4}{x^2 - 16}$ by $\dfrac{x - 1}{x^3 - x^2 + 4\,x - 4}$.

17. $\dfrac{x^3 - 7\,x + 6}{x^4 - 3\,x^3 - 7\,x^2 + 27\,x - 18}$ by $\dfrac{x^3 + 3\,x^2 - x - 3}{x^4 - 10\,x^2 + 9}$.

18. $\dfrac{2\,x^4 + 3\,x^3 - 14\,x^2 - 9\,x + 18}{x^4 + x^3 - 7\,x^2 - 13\,x - 6}$ by $\dfrac{2\,x^4 - 9\,x^3 + 4\,x^2 + 21\,x - 18}{x^4 + 6\,x^3 + 13\,x^2 + 12\,x + 4}$.

SIMPLIFICATION OF COMPLEX FRACTIONS

142. Prob. *To reduce a complex to a simple fraction.*

RULE. *Multiply numerator and denominator of the complex fraction by the* **l. c. m.** *of the denominators of the partial fractions.*

DEM. This does not change the value of the complex fraction (Art. 127), and it removes the partial denominators, since they are factors of the quantity by which all the partial fractions are multiplied.

143. NOTE. A complex fraction may be simplified also by uniting the terms of the numerator and denominator separately, and then dividing the former result by the latter.

EXAMPLES XLI

Reduce the following to simple fractions:

1. $\dfrac{\dfrac{a}{b} - \dfrac{c}{d}}{\dfrac{e}{f} + \dfrac{g}{h}}$.

2. $\dfrac{\dfrac{a}{b} - \dfrac{b}{a}}{\dfrac{a+b}{ab}}$.

3. $\dfrac{a+b+\dfrac{b^2}{a}}{a+b+\dfrac{a^2}{b}}$.

4. $\dfrac{\dfrac{a+1}{a-1}+\dfrac{a-1}{a+1}}{\dfrac{a+1}{a-1}-\dfrac{a-1}{a+1}}$.

5. $\dfrac{\dfrac{x}{x+y}-\dfrac{x-y}{x}}{\dfrac{x}{x-y}-\dfrac{x+y}{x}}$.

6. $\dfrac{a+6-\dfrac{3}{a+4}}{a+1-\dfrac{18}{a+4}}$.

7. $\dfrac{\dfrac{a}{bc}+\dfrac{b}{ac}+\dfrac{c}{ab}}{\dfrac{ab}{c}+\dfrac{ac}{b}+\dfrac{bc}{a}}$.

8. $\dfrac{\dfrac{m-n}{m+n}+\dfrac{m^2+n^2}{m^2-n^2}}{\dfrac{m^2}{m-n}+\dfrac{m^2n+n^3}{(m-n)^2}}$.

144. Prob. *To simplify a terminated continued fraction.*

Rule. *Begin at the bottom and simplify backwards and upwards, performing, step by step, the operations indicated.*

EXAMPLES XLII

Simplify the following:

1. $\dfrac{1}{x+\dfrac{1}{y+\dfrac{1}{z}}}$.

2. $\dfrac{a}{b+\dfrac{c}{d+\dfrac{e}{f}}}$.

3. $\dfrac{x}{x-\dfrac{x+2}{x+2-\dfrac{x+1}{x}}}$.

4. $\dfrac{1}{a^2-\dfrac{a^3-1}{a+\dfrac{1}{a+1}}}$.

5. $\dfrac{a+\dfrac{b}{1+\dfrac{a}{b}}}{a-\dfrac{b}{1-\dfrac{a}{b}}}(a^6-b^6)$.

6. $\dfrac{1}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}$.

145. Prob. *To free a fraction from negative exponents.*

Rule. *If the quantity affected with a negative exponent is a factor of the numerator, transfer it to the denominator, or if a*

factor of the denominator, transfer it to the numerator, and change the sign of the exponent.

If it is a term or part of a term of a polynomial, transfer it to the denominator of the term in which it stands, changing the sign of the exponent, and then reducing the resulting complex fraction to a simple one; or, what is the same thing, *without this transference multiply both numerator and denominator by the same quantity affected with a numerically equal positive exponent.*

Dem. The rule is a direct consequence of the signification of a negative exponent (Art. 9), viz. the reciprocal of what the expression would be if the exponent were positive.

EXAMPLES XLIII

Free the following from negative exponents:

1. $\dfrac{a^{-2}b^{-3}c^4}{x^{-1}y^{-n}}$.

2. $\dfrac{a^{-2}-b^{-3}}{a^{-3}+b^{-2}}$.

3. $\dfrac{(x-y)^{-3}}{x^{-3}+y^{-3}}$.

4. $\dfrac{xy^{-3}(a^{-2}-b^{-2})}{z^{-2}(a^{-3}+b^{-3})}$.

5. $\dfrac{(x^{-2}+y^{-1})^{-3}}{(x^{-1}-y^{-2})^{-2}}$.

6. $\dfrac{x^{-m}(x^2-y)^{-n}}{(x^{-2}-1)^{-p}}$.

CHAPTER IX

THEORY OF EXPONENTS, INVOLUTION, AND EVOLUTION

SECTION I — THEORY OF EXPONENTS

146. We may either define fractional and negative exponents, as in Art. 9, and prove the index law of multiplication to be general, as in Art. 48, or we may assume the index law as proved for positive integral exponents to be general, and determine what fractional and negative exponents should mean.

147. Theorem. *If the index law of multiplication is made general,*

1st. The numerator of a fractional exponent denotes a power, and the denominator a root.

2d. A negative exponent denotes the reciprocal of what it would denote if positive.

DEM. 1st. By definition (Art. 9, 2d (a)),

$$(x^{\frac{m}{n}})^n = x^{\frac{m}{n}} \cdot x^{\frac{m}{n}} \cdot x^{\frac{m}{n}} \cdots \text{ to } n \text{ factors.} \tag{1}$$

By the index law of multiplication (Art. 48, 1st) this is

$$x^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}+\cdots \text{ to } n \text{ terms}} = x^m. \tag{2}$$

Therefore,
$$(x^{\frac{m}{n}})^n = x^m. \tag{3}$$

Extracting the nth root of these equals, we have

$$x^{\frac{m}{n}} = \sqrt[n]{x^m}. \tag{4}$$

2d. By the index law of multiplication (Art. 48),

$$x^n \cdot x^{-n} = x^0 = 1 \text{ (Art. 70).}$$

Dividing these equals by x^n, we have

$$x^{-n} = \frac{1}{x^n}.$$

EXAMPLES XLIV

Express the following without fractional and negative exponents:

1. $a^{\frac{3}{2}}b^{\frac{2}{3}}$.   2. $a^{\frac{1}{2}}x^{\frac{2}{3}}$.   3. $5\,(ax)^{-\frac{3}{5}}$.

4. $\dfrac{2}{3\,x^{-\frac{1}{2}}}$.   5. $\dfrac{5\,x^{-\frac{3}{4}}}{7\,y^{\frac{2}{3}}}$.   6. $\dfrac{7^{\frac{1}{2}}\,x^{-\frac{1}{n}}}{6^{\frac{1}{3}}\,y^{\frac{p}{q}}}$.

7. $\dfrac{3\,a^2b^{-2}}{5^{\frac{1}{2}}\,a^{-\frac{3}{2}}b^{\frac{1}{3}}}$.   8. $\dfrac{5^{-2}\,x^{\frac{3}{5}}y^{-\frac{2}{3}}}{x^{-\frac{3}{5}}y^{\frac{2}{3}}}$.   9. $\dfrac{x^{\frac{m}{n}}y^{-p}}{4^{-\frac{1}{3}}\,y^{\frac{1}{n}}}$.

Express the following without radical signs and negative exponents:

10. $\sqrt{x^3}\,\sqrt[3]{y^2}$.   11. $5\sqrt[5]{x^{10}}\,\sqrt[4]{y^2}$.   12. $\sqrt{7}\,\sqrt[3]{x^{\frac{1}{2}}}\,\sqrt[3]{y^{-2}}$.

13. $\dfrac{\sqrt[3]{6}\,\sqrt{x^{-5}}\,\sqrt[5]{y^2}}{\sqrt{x}\,\sqrt[5]{y^{-2}}}$.   14. $\dfrac{a\sqrt{x^3}\,\sqrt[3]{y^{-2}}}{\sqrt{5}\,\sqrt[4]{a}\,\sqrt{x^{-3}}\,\sqrt[3]{y^2}}$.

15. $\dfrac{\sqrt[m]{x^s}\,\sqrt[n]{y^t}}{\sqrt[m]{x^{t}}\,\sqrt[n]{y^{-s}}}$.   16. $\sqrt[3]{(a+b)^2}\,\sqrt{(a-b)^{-3}}$.

17. $\dfrac{\sqrt[3]{(a^2+2\,b)^4}}{\sqrt{(x+y)^{-3}}}$.   18. $\dfrac{\sqrt[m]{(x^2-y^2)^{-p}}}{\sqrt[n]{(x^2+y^2)^{-q}}}$.

148. Theorem. *The product of several quantities affected collectively by an exponent is equal to the product of the several quantities affected separately by the same exponents.*

DEM. 1st. When the exponent is a positive integer.

$$\begin{aligned}
&(abc\cdots)^n \\
&\quad=(abc\cdots)(abc\cdots)(abc\cdots)\cdots \text{ to } n \text{ factors,} \\
&\quad=(aaa\cdots \text{ to } n \text{ factors})(bbb\cdots \text{ to } n \text{ factors})(ccc\cdots \text{ to } n \text{ factors})\cdots, \\
&\quad= a^nb^nc^n\cdots.
\end{aligned}$$

2d. When the exponent is a positive fraction.

$$\begin{aligned}(a^{\frac{m}{n}}b^{\frac{m}{n}}c^{\frac{m}{n}}\cdots)^n &= (a^{\frac{m}{n}})^n(b^{\frac{m}{n}})^n(c^{\frac{m}{n}})^n \cdots \text{ by the 1st case,}\\ &= a^m b^m c^m \cdots \text{ by (3) of Art. 147,}\\ &= (abc\cdots)^m \text{ by the 1st case.}\end{aligned}$$

Extracting the nth root of these equals, we have

$$a^{\frac{m}{n}}b^{\frac{m}{n}}c^{\frac{m}{n}}\cdots = (abc\cdots)^{\frac{m}{n}}.$$

3d. When the exponent is negative.

By Art. 71, and the 1st case of this demonstration,

$$(abc\cdots)^{-n} = \frac{1}{(abc\cdots)^n} = \frac{1}{a^n b^n c^n \cdots} = a^{-n}b^{-n}c^{-n}\cdots.$$

149. Cor. By the proposition,

$$\left(\frac{a}{b}\right)^n = \left(a \times \frac{1}{b}\right)^n = a^n \times \frac{1}{b^n} = \frac{a^n}{b^n};$$

also, $$\sqrt[n]{\frac{a}{b}} = \sqrt[n]{a \times \frac{1}{b}} = \sqrt[n]{a} \times \frac{1}{\sqrt[n]{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

150. Sch. The above theorem and corollary are briefly stated thus: *The power of the product equals the product of the powers, and the root of the product equals the product of the roots* (the indices being the same); *also the power of the quotient equals the quotient of the powers, and the root of the quotient equals the quotient of the roots* (the indices being the same).

The student needs to be cautioned against supposing that the power or root of the sum or difference equals the sum or difference of the powers or roots.

151. Prob. *To affect a monomial with any exponent.*

Rule. *Multiply the exponent of each of the factors by the given exponent.*

Dem. Let it be required to affect x^m with the exponent n, m having any value, integral or fractional, positive or negative.

1st. When n is a positive integer.

$$(x^m)^n = x^m x^m x^m \cdots \text{ to } n \text{ factors},$$
$$= x^{m+m+m+\cdots \text{ to } n \text{ terms}},$$
$$= x^{mn}.$$

2d. When n is a positive fraction, as $\frac{p}{q}$.

By Art. 147, $(x^m)^{\frac{p}{q}}$ is the qth root of the pth power of x^m, *i.e.*,

$$(x^m)^{\frac{p}{q}} = \sqrt[q]{(x^m)^p},$$
$$= \sqrt[q]{x^{mp}}, \text{ by the 1st case of this demonstration,}$$
$$= x^{\frac{mp}{q}}, \text{ by Art. 147.}$$

3d. When n is negative.

By Art. 147, and the 1st case of this demonstration,

$$(x^m)^{-n} = \frac{1}{(x^m)^n} = \frac{1}{x^{mn}} = x^{-mn}.$$

Finally, by Art. 148, what has been proved above for x^m applies to any number of factors of any monomial.

EXAMPLES XLV

Perform the following indicated operations:

1. $(a^3b^2)^2$.
2. $(a^2b^{-2}c^{\frac{1}{2}})^2$.
3. $(a^{\frac{3}{2}}b^{-\frac{1}{4}}c^{\frac{2}{3}})^2$.

4. $(3^2x^3y^{-\frac{2}{3}})^3$.
5. $(5x^{\frac{1}{2}}y^{\frac{3}{4}}z^{-1})^4$.
6. $(6^{\frac{1}{2}}x^2y^{-\frac{2}{5}}z^{\frac{4}{5}})^5$.

7. $\left(\dfrac{a^3}{b^2c^3}\right)^2$.
8. $\left(\dfrac{ax^{\frac{1}{3}}}{b^2y^{\frac{4}{3}}}\right)^3$.
9. $\left(\dfrac{3ax^{\frac{3}{2}}}{5by^{\frac{5}{4}}}\right)^4$.

10. $(ax^3y^6)^{\frac{2}{3}}$.
11. $(25a^4x^6y^{-2})^{\frac{1}{2}}$.
12. $\left(\dfrac{8a^3x^6}{27by^3}\right)^{\frac{1}{3}}$.

13. $(5x^{-2}y^3)^{-3}$.
14. $(a^{-\frac{3}{2}}x^{-3}y^6)^{-\frac{2}{3}}$.
15. $(32a^5b^{-10}c^{\frac{5}{2}})^{\frac{3}{5}}$.

16. $(-64x^6y^3z^{3m})^{\frac{2}{3}}$.
17. $\left\{\sqrt[4]{(x^{-\frac{2}{3}}y^{\frac{1}{2}})^3}\right\}^{-\frac{2}{3}}$.
18. $\left(\sqrt[m]{x^{mq}y^{\frac{q}{p}}z^{-q}}\right)^{\frac{p}{q}}$.

SECTION II — INVOLUTION

152. Involution is the process of raising quantities to required powers.

153. Prob. *To raise a monomial to any required power.*

RULE. *Use the numerical coefficient as many times as a factor as there are units in the degree of the power, and multiply the exponent of each literal factor by the exponent of the required power, writing the minus sign before the result only when an odd power of a negative quantity is required.*

This is but an application of the definition of a power (Art. 9), affecting a monomial with any exponent (Art. 151), and the law of signs in multiplication (Art. 47).

154. Factorial *n* is the product of all the integral numbers from 1 to n inclusive, and is written $\underline{\lfloor n}$.*

Thus, $\underline{\lfloor 3} = 2 \cdot 3$, $\underline{\lfloor 5} = 2 \cdot 3 \cdot 4 . 5$, $\underline{\lfloor n} = 2 \cdot 3 \cdot 4 \cdots n$.

155. Prob. *To raise a binomial to any required power.*

FORMULA. *x, y, and m being any quantities whatever, positive or negative, integral or fractional,*

$$(x+y)^m = x^m + mx^{m-1}y + \frac{m(m-1)}{\underline{\lfloor 2}}x^{m-2}y^2 + \frac{m(m-1)(m-2)}{\underline{\lfloor 3}}x^{m-3}y^3$$
$$+ \frac{m(m-1)(m-2)(m-3)}{\underline{\lfloor 4}}x^{m-4}y^4 + \text{etc.}$$

This is *Newton's Binomial Formula*, the demonstration of which is given in a subsequent part of the work. From an inspection of the formula we deduce the following theorem, called

THE BINOMIAL THEOREM

In the expansion of a binomial affected with any exponent, the exponent of the first letter begins in the first term with the exponent

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* The notation $n!$ is also employed.

of the binomial, and in each succeeding term decreases by 1; while the exponent of the second letter begins in the second term with 1, and in each succeeding term increases by 1.

The coefficient of the first term is 1; that of the second term is the exponent of the binomial; and if the coefficient of any term be multiplied by the exponent of the first letter in that term and divided by the exponent of the second letter increased by 1, the result will be the coefficient of the next term.

156. Cor. 1. *The expansion of a binomial terminates when m is a positive integer, and the number of terms is $m+1$.*

For the coefficient $m(m-1)(m-2)\cdots(m-m)$, which is in the $(m+2)$th term, and all subsequent coefficients, are 0.

157. Cor. 2. *When m is a positive integer, the coefficients equally distant from the extremes are numerically equal.*

For the expansion of $(b+a)^m$ has the same value as the expansion of $(a+b)^m$, but the terms occur in the reverse order. Hence beyond the middle, which is a term or a sign, according as m is even or odd, the coefficients need not be computed.

158. Cor. 3. *If the sign of the second letter is minus, and m is a positive integer, the signs of the terms of the expansion will be alternately plus and minus.*

For the odd powers of a minus quantity are minus.

159. Note. At this stage of his progress the student should learn the above theorem and familiarize himself with its use in raising binomials to powers, *i.e.* for positive, integral exponents. He should be careful to note that, while such a form as $(3a^2-2b^3)^5$ may be expanded by the theorem, the laws apply, not to the exponents and coefficients of a and b in the full expansion, but to those of $3a^2$ and $-2b^3$, regarded as the terms of the binomial. These should be kept in parentheses through the expansion, the indicated operations being performed afterward. It must not be forgotten that an odd power of a negative quantity is plus and an even power minus.

EXAMPLES XLVI

Expand the following by the Binomial Theorem:

1. $(3x^2-2y^3)^5$.

Solution. Here $3x^2$ takes the place of the first letter, and $-2y^3$ of the second letter of the formula.

Hence we have by the theorem,

$$\begin{aligned}(3x^2-2y^3)^5 &= (3x^2)^5+5(3x^2)^4(-2y^3)+10(3x^2)^3(-2y^3)^2\\ &\quad +10(3x^2)^2(-2y^3)^3+5(3x^2)(-2y^3)^4+(-2y^3)^5\\ &= 243x^{10}-810x^8y^3+1080x^6y^6-270x^4y^9+240x^2y^{12}-32y^{15}\end{aligned}$$

Since $m=5$, the expansion will contain 6 terms (Art. 156), and the coefficients after the middle sign are the same as those before it, but in the reverse order, and may be written without further computation (Art. 157).

2. $(x+y)^4$. **3.** $(a-b)^5$. **4.** $(2x^2-y)^3$.

5. $(1-x^2)^4$. **6.** $(a+b)^6$. **7.** $(x^2-y^2)^4$.

8. $(x^{\frac{1}{2}}+2y)^4$. **9.** $(1+2x^2)^5$. **10.** $(a^{\frac{1}{2}}+2b^2x^{-1})^4$.

11. $(x-\frac{1}{2})^6$. **12.** $(\sqrt[4]{a}+\sqrt{b})^8$. **13.** $(\frac{1}{2}x^{-2}-2a^2y^3)^6$.

14. $(x-y+z)^4$. **15.** $(\sqrt{x+1}-\sqrt{x-1})^4$. **16.** $(x+\sqrt{x^2-1})^6$.

160. A special theorem for squaring a polynomial has been given in Art. 64. It is convenient to have also a special theorem for cubing a polynomial.

161. Theorem. *The cube of any polynomial is the algebraic sum of the cubes of all the terms, three times the square of each term into each of the other terms, and six times each group that can be formed with three terms each as factors.*

Dem. When there are more than two terms, part of them may be treated as constituting the first term, and the others as the second term of a binomial, thus:

$$\begin{aligned}(a+b+c)^3 &= [(a+b)+c]^3=(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3\\ &= a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3\\ &= a^3+b^3+c^3+3a^2b+3a^2c+3b^2a+3b^2c+3c^2a+3c^2b+6abc.\end{aligned}$$

Again, $(a+b+c-d)^3=[(a+b)+(c-d)]^3$

$$=(a+b)^3+3(a+b)^2(c-d)+3(a+b)(c-d)^2+(c-d)^3$$

$$=a^3+b^3+c^3-d^3+3\,a^2b+3\,a^2c-3\,a^2d+3\,b^2a+3\,b^2c$$

$$-3\,b^2d+3\,c^2a+3\,c^2b-3\,c^2d+3\,d^2a+3\,d^2b+3\,d^2c$$

$$+6\,abc-6\,abd-6\,acd-6\,bcd.$$

By inspecting the results of these operations we deduce the important theorem stated above.

EXAMPLES XLVII

Cube each of the following:

1. $4\,x^2+2\,x-3.$

Operation. It is best first to write in order the different powers of x, and then supply the coefficients in the order of the rule, thus:

x^6	x^5	x^4	x^3	x^2	x	
64	96	-144	8	-36	54	-27
		48	-144	108		

$$64\,x^6+96\,x^5-96\,x^4-136\,x^3+72\,x^2+54\,x-27$$

2. $2\,x^2-3\,x+5.$

3. $6\,x^2-4\,xy+2\,y^2.$

4. $4\,x^3-2\,x^2+3\,x+5.$

5. $x^4-2\,x^3+3\,x^2-x+2.$

SECTION III — EVOLUTION

162. Evolution is the process of extracting roots of quantities.

In extracting any even root of a quantity there are always two results, differing only in sign, since even powers of both positive and negative quantities are plus. For the present purposes it is customary to write only the positive root, or in case of polynomials, the root that begins with a positive term.

163. Prob. *To extract any root of a monomial.*

Rule. *Extract the required root of the coefficient and divide the exponent of each letter by the index of the root.*

Dem. This is but an application of Art. 151, since to extract the square root is to affect a quantity with the exponent $\frac{1}{2}$, the cube root $\frac{1}{3}$, the nth root $\frac{1}{n}$, etc.

164. Prob. *To extract a root whose index is a composite number.*

Rule. *Extract successively the roots indicated by the prime factors of the index.*

Dem. By Arts. 147 and 151,

$$\sqrt[mn]{x} = x^{\frac{1}{mn}} = (x^{\frac{1}{m}})^{\frac{1}{n}} = \sqrt[n]{\sqrt[m]{x}};$$

also $$\sqrt[mn]{x} = x^{\frac{1}{mn}} = (x^{\frac{1}{n}})^{\frac{1}{m}} = \sqrt[m]{\sqrt{x}}.$$

Hence the 4th root is the square root of the square root; the 6th root is the cube root of the square root, or the square root of the cube root; the 8th root is the square root of the square root of the square root; the 9th root is the cube root of the cube root; and so on.

SQUARE ROOT OF POLYNOMIALS

Case I

165. *When the square is a trinomial.*

Rule. *Arrange with reference to one letter, extract the square root of the extreme terms, and connect the results by the sign of the middle term.*

This follows from Arts. 61 and 62. The expression is not a perfect square unless the middle term is twice the product of the square roots of the extreme terms.

Case II

166. *When the square contains only two powers of any one of its letters.*

Rule. *Arrange with reference to any letter which has only two powers, regard the terms not containing that letter as the third term of a trinomial, and proceed as in the preceding case.*

EXAMPLES XLVIII

Find the square root of each of the following:

1. $9a^4 + 4b^2 + c^6 + 12a^2b - 6a^2c^3 - 4bc^3$.

SOLUTION. Here we have but two powers of each letter, viz., a^4 and a^2, b^2 and b, c^6 and c^3. Arranging with reference to a (any other letter would do as well), we have

$$9a^4 + (12b - 6c^3)a^2 + (4b^2 - 4bc^3 + c^6).$$

The square root of the first term is $3a^2$, and of the last term $2b - c^3$; while the middle term is *plus* twice the product of these square roots. Hence the square root is $3a^2 + 2b - c^3$.

2. $a^2 + b^2c^2d^2 - 2ab^2c^2d^3 + b^4c^4d^6 + 2abcd - 2b^3c^3d^4$.

SOLUTION. Here a is the only letter having but two powers. Arranging with reference to this, we have

$$a^2 + (2bcd - 2b^2c^2d^3)a + (b^2c^2d^2 - 2b^3c^3d^4 + b^4c^4d^6)$$

or

$$a^2 + (2bcd - 2b^2c^2d^3)a + (bcd - b^2c^2d^3)^2.$$

Hence the square root is $a + bcd - b^2c^2d^3$.

3. $4a^2 + 9b^4 - 12ab^2 - 24b^2d^3 + 16ad^3 + 16d^6$.

4. $16x^4 + 16x^2z + y^2 - 8x^2y + 4z^2 - 4yz$.

5. $25x^4 + 20x^2y - 4yz^2 + 4y^2 + z^4 - 10x^2z^2$.

6. $9x^6 + z^2 + 12x^3y^2 - 4y^2z + 4y^4 - 6x^3z$.

7. $16a^4 + 24a^2b + 9b^2 - 16a^2c^3 + 4c^6 + 8a^2d + d^2 - 12bc^3 + 6bd - 4c^3d$.

SOLUTION. Here each letter has but two powers. Arranging with reference to a and, at the same time, arranging with reference to b the terms that do not contain a, we have

$$16a^4 + (24b - 16c^3 + 8d)a^2 + [9b^2 + (6d - 12c^3)b + (4c^6 - 4c^3d + d^2)].$$

The square root of the first term being $4a^2$ and of the last term (the part in the brackets) $3b - 2c^3 + d$, and the middle term *plus* twice the product of these square roots, the square root of the whole expression is

$$4a^2 + 3b - 2c^3 + d.$$

8. $9a^6 - 6a^3b^2 - 12a^3c + 24a^3d + b^4 + 4b^2c - 8b^2d + 4c^2 - 16cd + 16d^2$.

9. $36a^6 + 36a^3c - 24a^3b^2 - 60a^3d^4 + 4b^4 + 20b^2d^4 - 12b^2c + 9c^2 - 30cd^4 + 25d^8$.

10. $125a^9 + 225a^6b^2 - 150a^6c + 135a^3b^4 - 180a^3b^2c + 60a^3c^2 + 27b^6 - 54b^4c + 36b^2c^2 - 8c^3$.

Case III

167. *Any perfect square not included in Cases I and II.*

Solution. By Art. 64 we have, after arranging in descending powers of x,

$$(ax^n + bx^{n-1} + cx^{n-2} + dx^{n-3} + \cdots)^2$$
$$= a^2x^{2n} + 2abx^{2n-1} + (b^2 + 2ac)x^{2n-2} + (2bc + 2ad)x^{2n-3} + \cdots.$$

In squaring a polynomial of the nth degree a term in x^{2n} can result only from the square of the first term of the root; *hence if we take the square root of the first term of the arranged square, we shall have the first term of the square root.*

A term in x^{2n-1} can result only from twice the first term of the root by the second; *hence if we divide the second term of the arranged square by twice the first term of the root, we shall obtain the second term of the root.*

Terms in x^{2n-2} can result only from the square of the second term of the root and twice the first by the third; *hence if we subtract from the third term of the arranged square the square of the second term of the root and divide the remainder by twice the first term of the root, we shall obtain the third term of the root.*

Terms in x^{2n-3} can result only from twice the second term of the root by the third and twice the first by the fourth; *hence if we subtract from the fourth term of the arranged square twice the second term of the root by the third and divide the remainder by twice the first term of the root, we shall obtain the fourth term of the root.*

In the same way four terms may be obtained from the other end of the square, i.e., *obtained as they would be if the square were written the other end foremost, the signs being left temporarily ambiguous.*

In practice it is better to obtain terms alternately from the two ends, continuing the operation only until reaching a term that is numerically the same, from whichever end obtained. The sign of

this term as obtained from the first end will determine the signs of the terms obtained from the last end.

168. Sch. 1. If the square of the second term of the root is similar to the fourth term of the given polynomial instead of the third, the third term of the root is found by dividing the whole of the third term of the given polynomial by twice the first term of the root.

If the square of the second term of the root is similar to a missing term of the given polynomial between the second and third terms, the third term of the root is found by subtracting the square of the second term of the root from 0, the coefficient of the missing term, and dividing the remainder by twice the first term of the root.

169. Sch. 2. Test. If the overlapping term (*i.e.*, the term containing the same power of x from whichever end obtained) is not numerically the same when found from each end, the polynomial is not a perfect square. When it is the same, to ascertain whether the polynomial is a perfect square, it is necessary simply to note how those terms not used in the operation are made up from the square of the supposed square root.

170. Note. The advantage of this method over that employed for extracting the square root of numbers is its brevity, the square root being written at once, without writing any intermediate steps. The written work in the model solutions below is merely for explanation and can all be omitted in practice.

EXAMPLES XLIX

Find the square root of each of the following:

1. $25x^4 - 30x^3 + 49x^2 - 24x + 16.$

Solution. If this is a perfect square,

$$\text{first term of sq. rt.} = \sqrt{25x^4} = 5x^2,$$

$$\text{second term of sq. rt.} = -30x^3 \div 2(5x^2) = -3x,$$

$$\text{last term of sq. rt.} = \sqrt{16} = \pm 4,$$

$$\text{next to last term of sq. rt.} = -24x \div 2(\pm 4) = \mp 3x.$$

This term ($\mp 3x$) is numerically the same as the second, and to make it the same in sign the upper signs must be used. Hence the square root is $5x^2 - 3x + 4$.

TEST. In the operation all the terms of the given polynomial have been used except $49x^2$. By reference to Art. 64 it will be seen that in squaring $5x^2 - 3x + 4$, terms in x^2 can result only from the square of the second term and twice the first by the third. This gives $(-3x)^2 + 2(5x)(4) = 49x^2$, showing the given polynomial to be a perfect square.

2. $49x^4 + 28x^3 - 38x^2 - 12x + 9.$

3. $36x^6 - 36x^5 - 3x^4 + 6x^3 + x^2.$

4. $16 - 40x + 89x^2 - 80x^3 + 64x^4.$

5. $8x + 4 + x^4 + 4x^3 + 8x^2.$

6. $x^4 - x^3 + \frac{x^2}{4} + 4x - 2 + \frac{4}{x^2}.$

7. $36x^4 - 48x^3y + 52x^2y^2 - 24xy^3 + 9y^4.$

SOLUTION. If this is a perfect square,

$$\text{first term of sq. rt.} = \sqrt{36x^4} = 6x^2,$$

$$\text{second term of sq. rt.} = -48x^3y \div 2(6x^2) = -4xy,$$

$$\text{last term of sq. rt.} = \sqrt{9y^4} = \pm 3y^2,$$

$$\text{next to last term of sq. rt.} = -24xy^3 \div 2(\pm 3y^2) = \mp 4xy.$$

We see that there are but three terms and that the plus sign of $\sqrt{9y^4}$ must be used. Hence the square root is $6x^2 - 4xy + 3y^2$.

TEST. $(4xy)^2 + 2(6x^2)(3y^2) = 52x^2y^2.$

8. $25x^4 - 30x^3y - 31x^2y^2 + 24xy^3 + 16y^4.$

9. $9x^4 + 24x^3y - 14x^2y^2 - 40xy^3 + 25y^4.$

10. $16x^8 - 32x^6y^2 - 40x^4y^4 + 56x^2y^6 + 49y^8.$

11. $x^8 + 2x^5 + 2x^4 + x^2 + 2x + 1.$

12. $x^6 - 6x^5 + 9x^4 - 10x^3 + 30x^2 + 25.$

13. $49x^{10} - 56x^8 + 16x^6 + 126x^5 - 72x^3 + 81.$

14. $9x^6 + 6x^5 - 11x^4 + 20x^3 + 12x^2 - 16x + 16.$

SOLUTION. If this is a perfect square,

$$\text{first term of sq. rt.} = \sqrt{9\,x^6} = 3\,x^3,$$

$$\text{second term of sq. rt.} = 6\,x^5 \div 2(3\,x^3) = x^2,$$

$$\text{third term of sq. rt.} = [-11\,x^4 - (x^2)^2] \div 2(3\,x^3) = -2\,x,$$

$$\text{last term of sq. rt.} = \sqrt{16} = \pm 4,$$

$$\text{next to last term of sq. rt.} = -16\,x \div 2(\pm 4) = \mp 2\,x.$$

This term ($\mp 2\,x$) is numerically the same as the third, and to make it the same in sign the upper signs must be used. Hence the square root is $3\,x^3 + x^2 - 2\,x + 4$.

TEST. All the terms of the given polynomial have been used except $20\,x^3$ and $12\,x^2$. In squaring $3\,x^3 + x^2 - 2\,x + 4$, terms in x^3 can result only from twice the first by the fourth and twice the second by the third, giving

$$2(3\,x^2)(4) + 2(x^2)(-2\,x) = 20\,x^3.$$

Terms in x^2 can result only from the square of the third and twice the second by the fourth, giving

$$(-2\,x)^2 + 2(x^2)(4) = 12\,x^2.$$

Hence the given polynomial is a perfect square.

15. $9\,x^6 - 24\,x^5 + 28\,x^4 - 46\,x^3 + 44\,x^2 - 20\,x + 25.$

16. $16\,x^6 - 40\,x^5 + 41\,x^4 - 44\,x^3 + 34\,x^2 - 12\,x + 9.$

17. $49\,x^6 + 42\,x^5 - 19\,x^4 - 68\,x^3 - 20\,x^2 + 16\,x + 16.$

18. $36\,x^6 - 48\,x^5y - 20\,x^4y^2 + 84\,x^3y^3 - 31\,x^2y^4 - 30\,xy^5 + 25\,y^6.$

19. $64\,x^6 - 16\,x^5 - 47\,x^4 - 90\,x^3 + 21\,x^2 + 36\,x + 36.$

20. $121\,x^6 - 66\,x^5y + 119\,x^4y^2 + 168\,x^3y^3 - 29\,x^2y^4 + 90\,xy^5 + 81\,y^6.$

21. $9\,x^6 + 6\,x^5y - 11\,x^4y^2 + 20\,x^3y^3 + 12\,x^2y^4 - 16\,xy^5 + 16\,y^6.$

22. $4\,x^6 + 16\,x^5 - 40\,x^3 + 16\,x + 4.$

SOLUTION. If this is a perfect square,

$$\text{first term of sq. rt.} = \sqrt{4\,x^6} = 2\,x^3,$$

$$\text{second term of sq. rt.} = 16\,x^5 \div 2(2\,x^3) = 4\,x^2,$$

$$\text{third term of sq. rt.} = [0 - (4\,x^2)^2] \div 2(2\,x^3) = -4\,x,$$

$$\text{last term of sq. rt.} = \sqrt{4} = \pm 2,$$

$$\text{next to last term of sq. rt.} = 16\,x \div 2(\pm 2) = \pm 4\,x.$$

This term ($\pm 4x$) is numerically the same as the third, and to make it the same in sign the lower signs must be used. Hence the square root is $2x^3 + 4x^2 - 4x - 2$.

Test. All the terms of the given polynomial have been used except $-40x^3$ and a missing term in x^2. In squaring $2x^3 + 4x^2 - 4x - 2$, terms in x^3 can result only from twice the first by the fourth and twice the second by the third, giving

$$2(2x^3)(-2) + 2(4x^2)(-4x) = -40x^3.$$

Terms in x^2 can result only from the square of the third term and twice the second by the fourth, giving

$$(-4x)^2 + 2(4x^2)(-2) = 0.$$

Hence the given polynomial is a perfect square.

23. $x^8 + 2x^6 + 2x^5 + 3x^4 + 2x^3 + 3x^2 + 2x + 1.$

24. $25x^8 + 20x^7 - 6x^6 - 4x^5 + 61x^4 + 24x^3 - 12x^2 + 36.$

25. $16x^{10} - 8x^8 + 16x^7 + 17x^6 - 28x^5 + 14x^3 - 8x^2 - 12x + 9.$

Solution. If this is a perfect square,

$$\text{first term of sq. rt.} = \sqrt{16x^{10}} = 4x^5,$$

$$\text{second term of sq. rt.} = -8x^8 \div 2(4x^5) = -x^3,$$

$$\text{third term of sq. rt.} = 16x^7 \div 2(4x^5) = 2x^2,$$

$$\text{last term of sq. rt.} = \sqrt{9} = \pm 3,$$

$$\text{next to last term of sq. rt.} = -12x \div 2(\pm 3) = \mp 2x,$$

$$\text{third from last term of sq. rt.} = [-8x^2 - (\mp 2x)^2] \div 2(\pm 3) = \mp 2x^2.$$

This term ($\mp 2x^2$) is numerically the same as the third, and to make it the same in sign the lower signs must be used. Hence the square root is $4x^5 - x^3 + 2x^2 + 2x - 3$.

As the square of the second term of the root is similar, not to the third term of the given polynomial, but to the fourth, the whole of the third term must be divided by twice the first term of the root.

26. $x^8 - 6x^7 + 5x^6 + 22x^5 - 18x^4 - 44x^3 + 9x^2 + 40x + 16.$

27. $25x^{10} - 30x^8 + 20x^7 - 31x^6$

$$+ 48x^5 + 28x^4 - 52x^3 + 40x^2 - 48x + 36.$$

28. $16x^{12} - 48x^{10} + 16x^9 + 12x^8$

$$- 24x^7 + 112x^6 - 12x^5 - 99x^4 + 36x^3 - 54x^2 + 81.$$

29. $16x^{10} - 40x^9 + 41x^8 + 4x^7$

$$- 74x^6 + 96x^5 - 45x^4 - 24x^3 + 54x^2 - 36x + 9.$$

SOLUTION. If this is a perfect square,

$$\text{first term of sq. rt.} = \sqrt{16\,x^{10}} = 4\,x^5,$$
$$\text{second term of sq. rt.} = -\,40\,x^9 \div 2(4\,x^5) = -\,5\,x^4,$$
$$\text{third term of sq. rt.} = [41\,x^8 - (-\,5\,x^4)^2] \div 2(4\,x^5) = 2\,x^3,$$
$$\text{fourth term of sq. rt.} = [4\,x^7 - 2(-\,5\,x^4)(2\,x^3)] \div 2(4\,x^5) = 3\,x^2,$$
$$\text{last term of sq. rt.} = \sqrt{9} = \pm\,3,$$
$$\text{next to last term of sq. rt.} = -\,36\,x \div 2(\pm\,3) = \mp\,6\,x,$$
$$\text{third from last term of sq. rt.} = [54\,x^2 - (\mp\,6\,x)^2] \div 2(\pm\,3\,x) = \pm\,3$$

This term $(\pm\,3\,x^2)$ is numerically the same as the fourth, and to make it the same in sign the upper signs must be used. Hence the square root is $4\,x^5 - 5\,x^4 + 2\,x^3 + 3\,x^2 - 6\,x + 3$.

30. $49\,x^{10} + 14\,x^9 - 27\,x^8 - 60\,x^7$
$\qquad - 46\,x^6 + 80\,x^5 + 38\,x^4 + 4\,x^3 - 31\,x^2 - 30\,x + 25.$

31. $16\,x^{10} - 16\,x^9 - 52\,x^8 + 52\,x^7$
$\qquad + 29\,x^6 - 102\,x^5 + 55\,x^4 + 106\,x^3 - 47\,x^2 + 16\,x + 64.$

32. $9\,x^{12} - 30\,x^{11} + 25\,x^{10} - 24\,x^8 + 52\,x^7$
$\qquad - 50\,x^6 + 50\,x^5 + 16\,x^4 - 16\,x^3 + 44\,x^2 - 20\,x + 25.$

33. $36\,x^{12} - 48\,x^{11} - 8\,x^{10} + 28\,x^9 + 56\,x^8 - 80\,x^7$
$\qquad - 79\,x^6 + 78\,x^5 + 47\,x^4 - 44\,x^3 - 61\,x^2 + 42\,x + 49.$

34. $49\,x^{14} + 70\,x^{13}y - 17\,x^{12}y^2 - 44\,x^{11}y^3 + 111\,x^{10}y^4 + 142\,x^9y^5$
$\qquad - 35\,x^8y^6 + 24\,x^7y^7 + 128\,x^6y^8 + 32\,x^5y^9 - 28\,x^4y^{10}$
$\qquad + 80\,x^3y^{11} + 52\,x^2y^{12} - 24\,xy^{13} + 36\,y^{14}.$

SUG. The signs of the last four terms can be determined by noting what sign of the fifth term of the root must be used in producing the fifth term of the given polynomial.

CUBE ROOT OF POLYNOMIALS

CASE I

171. *When the cube is a quadrinomial.*

RULE. *Arrange with reference to one letter and take the algebraic sum of the cube roots of the extreme terms.*

DEM. By the binomial formula or by actual multiplication we have

$$(a \pm b)^3 = a^3 \pm 3\,a^2b + 3\,ab^2 \pm b^3,$$

in which it is seen that the extreme terms of the quadrinomial are the cubes of the respective terms of the cube root.

Test. If the quadrinomial is a perfect cube, its second term will be three times the square of the first term of the root multiplied by the second, and its third term will be three times the first term of the root multiplied by the square of the second.

EXAMPLES L

Find the cube root of each of the following:

1. $8x^3 - 36x^2 + 54x - 27$.

Solution. If this is a perfect cube,

$$\text{first term of cu. rt.} = \sqrt[3]{8x^3} = 2x,$$

$$\text{second term of cu. rt.} = \sqrt[3]{-27} = -3.$$

Hence the cube root is $2x - 3$.

Test. We observe that

$$3(2x)^2(-3) = -36x^2,$$

and

$$3(2x)(-3)^2 = 54x.$$

2. $64x^6 + 96x^4y + 48x^2y^2 + 8y^3$.
3. $216x^3 + 540x^2 + 450x + 125$.
4. $125x^6 - 75x^4y^2 + 15x^2y^4 - y^6$.
5. $64x^3 - 144x^2y^2 + 108xy^4 - 27y^6$.

Case II

172. *When the cube contains only three powers of any one of its letters.*

Rule. *Arrange with reference to any one of the letters which has only three powers, regard the terms not containing that letter as the fourth term of a quadrinomial, and proceed as in the preceding case.*

EXAMPLES LI

Find the cube root of each of the following:

1. $8a^6 - 36a^4b + 54a^2b^2 - 27b^3 + 48a^4c^2 - 144a^2bc^2 + 108b^2c^2 + 96a^2c^4 - 144bc^4 + 64c^6$.

SOLUTION. Here each letter has but three powers. Arranging with reference to a, and, at the same time, arranging with reference to b the terms that do not contain a, we have

$$8a^6+(-36b+48c^2)a^4+(54b^2-144bc^2+96c^4)a^2+(-27b^3 \\ +108b^2c^2-144bc^4+64c^6).$$

Since this is a quadrinomial, if it is a perfect cube,

$$\text{first term of cu. rt.} = \sqrt[3]{8a^6} = 2a^2,$$

$$\text{second term of cu. rt.} = \sqrt[3]{-27b^3+108b^2c^2-144bc^4+64c^6} \\ = -3b+4c^2.$$

As the intermediate terms conform to the test named in Case I, the given polynomial is a perfect cube, and the cube root is $2a^2-3b+4c^2$.

2. $a^3+b^3+c^3+3a^2b+3a^2c+3ab^2+3ac^2+6abc+3b^2c+3bc^2$.

3. $27x^3+54x^2y-27x^2z^2+36xy^2-36xyz^2+9xz^4+8y^3-12y^2z^2$
$+6yz^4-z^6$.

4. $64x^6-96x^4y^3+144x^4z+48x^2y^6-144x^2y^3z+108x^2z^2-8y^9$
$+36y^6z-54y^3z^2+27z^3$.

5. $125w^3+150w^2x-225w^2y-300w^2z+60wx^2-180wxy$
$-240wxz+135wy^2+360wyz+240wz^2+8x^3-36x^2y$
$-48x^2z+54xy^2+144xyz+96xz^2-27y^3-108y^2z$
$-144yz^2-64z^3$.

CASE III

173. *Any perfect cube not included in Cases I and II.*

SOLUTION. By Art. 161 we have, after arranging in descending powers of x,

$$(ax^n+bx^{n-1}+cx^{n-2}+dx^{n-3}+\cdots)^3$$
$$=a^3x^{3n}+3a^2bx^{3n-1}+(3ab^2+3a^2c)x^{3n-2}+(b^3+6abc+3a^2d)x^{3n-3}+\cdots.$$

In cubing a polynomial of the nth degree, a term in x^{3n} can result only from the cube of the first term of the root; *hence, if we take the cube root of the first term of the arranged cube, we shall have the first term of the cube root.*

A term in x^{3n-1} can result only from three times the square of the first term of the root by the second; *hence, if we divide the second term of the arranged cube by three times the square of the first term of the root, we shall obtain the second term of the root.*

Terms in x^{3n-2} can result only from three times the first term of the root by the square of the second and three times the square of the first by the third; *hence, if we subtract from the third term of the arranged cube three times the first term of the root by the square of the second and divide the remainder by three times the square of the first term of the root, we shall obtain the third term of the root.*

Terms in x^{3n-3} can result only from the cube of the second term of the root, six times the product of the first three terms, and three times the square of the first term by the fourth; *hence, if we subtract from the fourth term of the arranged cube the cube of the second term of the root and six times the product of the first three terms of the root, and then divide the remainder by three times the square of the first term of the root, we shall obtain the fourth term of the root.*

In the same way four terms of the root may be obtained from the other end of the cube, i.e. *obtained as they would be if the cube were written the other end foremost.*

In practice, it is better to obtain terms alternately from the two ends, continuing the operation only until the terms meet in the middle of the root.

174. Sch. 1. There are the same kinds of exceptions here as noted in Sch. 1 of Case III for square root.

175. Sch. 2. Test. To ascertain whether the given polynomial is a perfect cube, it is necessary simply to note how those terms not used in the operation are made up from the cube of the supposed cube root.

176. Note. It is to be observed that, as in square root of polynomials, the cube root can be written at once, without writing any intermediate steps.

EXAMPLES LII

Find the cube root of each of the following:

1. $64x^6 + 96x^5 - 96x^4 - 136x^3 + 72x^2 + 54x - 27.$

SOLUTION. If this is a perfect cube,

$$\text{first term of cu. rt.} = \sqrt[3]{64x^6} = 4x^2,$$

$$\text{second term of cu. rt.} = 96x^5 \div 3(4x^2)^2 = 2x,$$

$$\text{last term of cu. rt.} = \sqrt[3]{-27} = -3.$$

There can be no other terms in the cube root, as there can be no terms between $2x$ and -3. Moreover, next to the last term of the cube root is $54x \div 3(-3)^2 = 2x$, the same as already obtained for the second term. Hence the cube root is $4x^2 + 2x - 3$.

TEST. In the operation all the terms of the given polynomial have been used except the third, the fourth, and the fifth. By reference to Art. 161 it will be seen that in cubing $4x^2 + 2x - 3$, terms in x^4 can result only from three times the square of the first into the third and three times the square of the second into the first, giving

$$3(4x^2)^2(-3) + 3(2x)^2(4x^2) = -96x^4.$$

Terms in x^3 can result only from the cube of the second and six times the product of the three, giving

$$(2x)^3 + 6(4x^2)(2x)(-3) = -136x^3.$$

Terms in x^2 can result only from three times the square of the second into the third and three times the square of the third into the first, giving

$$3(2x)^2(-3) + 3(-3)^2(4x^2) = 72x^2.$$

Hence the polynomial is a perfect cube.

2. $8x^6 - 36x^5y + 114x^4y^2 - 207x^3y^3 + 285x^2y^4 - 225xy^5 + 125y^6.$

3. $8x^{12} - 36x^{10} + 66x^8 - 63x^6 + 33x^4 - 9x^2 + 1.$

4. $8 - 12x + 42x^2 - 61x^3 + 87x^4 - 105x^5 + 87x^6 - 66x^7 + 36x^8 - 8x^9.$

SOLUTION. If this is a perfect cube,

$$\text{first term of cu. rt.} = \sqrt[3]{8} = 2,$$

$$\text{second term of cu. rt.} = -12x \div 3(2)^2 = -x,$$

$$\text{last term of cu. rt.} = \sqrt[3]{-8x^9} = -2x^3,$$

$$\text{next to last term of cu. rt.} = 36x^8 \div 3(-2x^3)^2 = 3x^2.$$

As there can be no other powers of x between the extreme terms 2 and $-2x^3$, there are but four terms, and the cube root is $2 - x + 3x^2 - 2x^3$.

5. $8x^9 - 36x^8y + 66x^7y^2 - 87x^6y^3 + 105x\,y^4 - 87x^4y^5 + 61x^3y^6 - 42x^2y^7 + 12xy^8 - 8y^9$.

6. $64x^9 - 96x^8 + 192x^7 + 88x^6 - 96x^5 + 366x^4 + 147x^3 - 15x^2 + 225x + 125$.

7. $x^{12} - 6x^{11} + 21x^{10} - 47x^9 + 81x^8 - 108x^7 + 126x^6 - 117x^5 + 99x^4 - 61x^3 + 42x^2 - 12x + 8$.

Solution. If this is a perfect cube,

first term of cu. rt. $= \sqrt[3]{x^{12}} = x^4$,

second term of cu. rt. $= -6x^{11} \div 3(x^4)^2 = -2x^3$,

third term of cu. rt. $= [21x^{10} - 3(x^4)(-2x^3)^2] \div 3(x^4)^2 = 3x^2$,

last term of cu. rt. $= \sqrt[3]{8} = 2$,

next to last term of cu. rt. $= -12x \div 3(2)^2 = -x$.

There can be no other powers of x between the extreme terms x^4 and 2. Moreover, the third term from the last is $[42x^2 - 3(2)(-x)^2] \div 3(2)^2 = 3x^2$, the same as already obtained for the third term from the other end. Hence the cube root is $x^4 - 2x^3 + 3x^2 - x + 2$.

8. $64x^{12} + 144x^{11} + 12x^{10} - 261x^9 + 66x^8 + 387x^7 + 82x^6 - 171x^5 + 666x^4 + 513x^3 - 54x^2 - 324x + 216$.

9. $27x^{15} - 54x^{14} + 63x^{13} + 64x^{12} - 204x^{11} + 87x^{10} + 187x^9 - 414x^8 - 36x^7 + 433x^6 - 192x^5 - 192x^4 + 408x^3 + 165x^2 - 225x - 125$.

EXAMPLES LIII

Solve the following by the principle of Art. 164:

1. The 4th root of
$$16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4.$$

2. The 6th root of
$$a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6.$$

3. The 6th root of
$$729 - 2916x^2 + 4860x^4 - 4320x^6 + 2160x^8 - 576x^{10} + 64x^{12}.$$

4. The 8th root of $x^8 - 16x^7y + 112x^6y^2 - 448x^5y^3 + 1120x^4y^4 - 1792x^3y^5 + 1792x^2y^6 - 1024xy^7 + 256y^8$.

177. Prob. *To extract any root of any quantity.*

Solution. By the Binomial Theorem or by actual multiplication we have

$$(a+b)^2 = a^2 + (2a+b)b,$$

$$(a+b)^3 = a^3 + (3a^2 + 3ab + b^2)b,$$

$$(a+b)^5 = a^5 + (5a^4 + 10a^3b + 10a^2b^2 + 5ab^3 + b^4)b,$$

etc., etc., etc.

In reversing the process it is seen that the required root of the first term of the given polynomial (or of the first period of the given number) will be the first term (or figure) of the root.

Could we divide the rest of the polynomial (or number) by the part in the parenthesis, we should obtain b, the next term (or figure) of the root. Only the first term (or figure) of this divisor is known until b is found. In case of polynomials, however, only the first term of the divisor is needed (and in case of numbers this first figure is much the larger part, since it is tens with reference to b as units). After the second term (or figure) of the root is found, the divisor can be completed.

If the root contains more than two terms (or figures), we may form a new trial divisor by considering the whole of the root now found as the first term (or figure), and proceed as before.

It is assumed that the student is familiar with the reason, as given in Arithmetic, for pointing off the number into periods of as many figures each as indicated by the index of the root, beginning at the decimal point.

178. Note. As already shown, the square root and cube root of polynomials can be written at once by inspection. In the extraction of roots of numbers the work is greatly facilitated by the use of logarithms, as shown in a subsequent part of the work.

CHAPTER X

SURDS AND IMAGINARIES

SECTION I — SURDS

179. A Radical is an indicated root of a quantity.

180. A Surd, or Irrational Quantity, is a radical quantity whose indicated root cannot be exactly extracted.

181. A surd is *Quadratic,* or of the *Second Degree, Cubic,* or of the *Third Degree, Biquadratic,* or of the *Fourth Degree,* etc., according as its index is 2, 3, 4, etc.

182. Similar Surds are expressions containing the same surd factor.

Thus, $2\sqrt{3a}$, $5m\sqrt{3a}$, $(a^2 - m^2)\sqrt{3a}$ are similar surds.

183. To Rationalize an Expression is to free it from surds.

184. The Rationalizing Factor is the factor by which a surd must be multiplied to rationalize it.

185. To Rationalize the Denominator of a Fraction is to free the denominator from surds without changing the value of the fraction.

186. The Simplest Form of a Surd is the form in which the smallest possible integral number is left under the radical sign.

187. To Simplify a Surd is to reduce it to its simplest form.

REDUCTION OF SURDS

188. The substance of all demonstrations in Reductions is to show:

1st. That the operation does not change the value of the expression.

2d. That the operation produces the required form.

189. Prob. *To simplify a surd when the quantity under the radical sign contains a factor of which the indicated root can be taken.*

RULE. *Write the required root of this factor as the coefficient of the indicated root of the other factor.*

This is because the root of the product equals the product of the roots (Art. 150).

EXAMPLES LIV

Simplify the following:

1. $5a\sqrt{147x^2y^5}$.

OPERATION. $5a\sqrt{147x^2y^5} = 5a\sqrt{49x^2y^4 \times 3y} = 35axy^2\sqrt{3y}$.

2. $\sqrt{32}$. 3. $\sqrt{75}$. 4. $\sqrt{72}$.

5. $2\sqrt{98}$. 6. $3\sqrt{125}$. 7. $4\sqrt{180}$.

8. $\sqrt{48a^2b^3}$. 9. $\sqrt{63a^5b^4}$. 10. $2c\sqrt{125ab^3c^5}$.

11. $4\sqrt{108x^2y^4}$. 12. $3a\sqrt{245b^3x^5}$. 13. $6a^2\sqrt{200m^3x^4}$.

14. $\sqrt[3]{56a^3b^5c^6}$. 15. $3\sqrt[3]{81x^3y^4}$. 16. $5\sqrt[3]{135ab^4c^5}$.

17. $a\sqrt{432a^{3m}b^{4n}}$. 18. $5b\sqrt[3]{128a^{4m}b^{3n+6}}$. 19. $7\sqrt[3]{-108a^6b^7c^9}$.

20. $2\sqrt{343x^4y^6}$. 21. $3\sqrt{720a^2b^{-4}c^5}$. 22. $5\sqrt[3]{448a^3b^{-6}c^7}$.

23. $\sqrt{a^3 + a^2b}$. 24. $\sqrt{18x^3 - 27x^2y}$.

25. $\sqrt{20a^5b^2 - 32a^6b^3}$. 26. $\sqrt{2x^4 + 4x^3y + 2x^2y^2}$.

27. $\sqrt{3x^5 - 6x^4y^2 + 3x^3y^4}$. 28. $\sqrt[3]{24a^4 - 32a^3b}$.

190. Prob. *To simplify a surd when the quantity under the radical sign is a perfect power of the degree indicated by a factor of the index.*

RULE. *Extract the root indicated by this factor of the index and write the result as a surd whose index is the other factor.*

This rule is simply an application of Art. 150.

EXAMPLES LV

Simplify the following:

1. $\sqrt[6]{343}$.

OPERATION. $\sqrt[6]{343} = \sqrt{\sqrt[3]{343}} = \sqrt{7}$.

2. $\sqrt[4]{49}$. **3.** $\sqrt[4]{169}$. **4.** $\sqrt[4]{225\,a^4b^2}$.

5. $\sqrt[6]{49}$. **6.** $\sqrt[6]{216}$. **7.** $\sqrt[6]{1000\,a^3b^6}$.

8. $\sqrt[8]{81}$. **9.** $\sqrt[8]{625\,a^8b^4}$. **10.** $\sqrt[9]{64}$.

11. $\sqrt[9]{-729\,a^9b^6c^3}$. **12.** $\sqrt[9]{1728\,a^{3m+6}}$. **13.** $2\sqrt[10]{32\,a^5b^{-10}}$.

14. $\sqrt[10]{196\,a^6b^4}$. **15.** $3\sqrt[12]{81\,a^4b^8}$. **16.** $\sqrt[12]{729\,a^6b^6}$.

17. $\sqrt[4]{16\,x^2 + 64\,xy + 64\,y^2}$.

18. $\sqrt[6]{64\,x^3 - 576\,x^2y + 1728\,xy^2 - 1728\,y^3}$.

191. Prob. *To reduce a rational quantity to the radical form.*

RULE. *Raise the quantity to the degree of the required radical and place the result under the corresponding radical sign.*

As the operation performed is the inverse of the operation indicated, the value of the quantity is not changed.

192. Prob. *To introduce under the radical sign the coefficient of a surd.*

RULE. *Raise it to the power of the same degree as the surd and multiply the result by the quantity under the radical sign.*

DEM. The coefficient being reduced by the last rule to the same form as the surd, we have the product of two like roots, which is equal to the same root of the product (Art. 150).

EXAMPLES LVI

1. Reduce to radicals of the 2d degree,

$$3\,ab^2,\ 7\,x^3y^2,\ 11\,xy^4,\ \text{and}\ 2\,x - 3\,y^2.$$

2. Reduce to radicals of the 3d degree,

$$4\,x^2y,\ 5\,a^2x^3,\ -3\,a^{-2}bc^2,\ \text{and}\ 6\,xy^2z^{-3}.$$

In the following introduce the coefficients under the radical sign or within the parenthesis:

3. $3\sqrt{2}$.   4. $4a\sqrt{3a}$.   5. $2\sqrt{\frac{1}{2}}$.

6. $\frac{1}{3}\sqrt{3}$.   7. $3\sqrt[3]{3}$.   8. $\frac{a}{5}\sqrt[3]{25}$.

9. $\frac{x}{4}\sqrt{\frac{128}{x^2}}$.   10. $(a+b)\sqrt{\frac{x}{a+b}}$.   11. $(a-b)\sqrt{\frac{1}{(a-b)^2}}$.

12. $(x-y)\sqrt{\frac{1}{x^2-y^2}}$.   13. $a^3\left(1-\frac{b^2}{a^2}\right)^{\frac{3}{2}}$.   14. $\frac{m+n}{m-n}\sqrt{\frac{m-n}{m+n}}$.

15. $8(x^2-y^2)^{\frac{3}{2}}$.   16. $x^6\left(1-\frac{1}{x^4}\right)^{\frac{3}{4}}$.   17. $(x-1)\sqrt{\frac{2}{x-1}+1}$.

193. Prob. *To reduce surds of different degrees to equivalent ones of the same degree.*

First Rule. *Write for the common index the* **l. c. m.** *of all the indices, and raise each quantity under the radical sign to a power whose degree is the factor by which its root index must be multiplied to produce this* **l. c. m.**

Second Rule. *Write the surds with fractional exponents, reduce these exponents to a common denominator, and express the results in the radical form.*

Dem. By either rule the value of each surd remains the same, since we extract a certain root and then raise the surd to the corresponding power.

194. Sch. Surds are readily compared in magnitude by first reducing them to equivalent surds of the same degree.

EXAMPLES LVII

Reduce the following to equivalent surds of the same degree:

1. $\sqrt{3x}$, $\sqrt[3]{2x^2}$, and $\sqrt[4]{5x^3}$.

OPERATION

$$\sqrt{3x} = \sqrt[12]{(3x)^6} = \sqrt[12]{729x^6},$$
$$\sqrt[3]{2x^2} = \sqrt[12]{(2x^2)^4} = \sqrt[12]{16x^8},$$
$$\sqrt[4]{5x^3} = \sqrt[12]{(5x^3)^3} = \sqrt[12]{125x^9}.$$

Or we may proceed thus:

$$\sqrt{3x} = 3^{\frac{1}{2}}x^{\frac{1}{2}} = 3^{\frac{6}{12}}x^{\frac{6}{12}} = \sqrt[12]{729x^6},$$

$$\sqrt[3]{2x^2} = 2^{\frac{1}{3}}x^{\frac{2}{3}} = 2^{\frac{4}{12}}x^{\frac{8}{12}} = \sqrt[12]{16x^8},$$

$$\sqrt[4]{5x^3} = 5^{\frac{1}{4}}x^{\frac{3}{4}} = 5^{\frac{3}{12}}x^{\frac{9}{12}} = \sqrt[12]{125x^9}.$$

2. $\sqrt{2}$ and $\sqrt[3]{3}$.

3. $\sqrt[3]{2}$ and $\sqrt[5]{2}$.

4. $\sqrt{2}$ and $\sqrt[5]{3}$.

5. $\sqrt[3]{4a^2}$ and $\sqrt[4]{3a^5}$.

6. $\sqrt[m]{a^3}$ and $\sqrt[n]{b^4}$.

7. $\sqrt[3]{a^{mr}}$ and $\sqrt[4]{b^n}$.

8. $\sqrt{a}$, $\sqrt[3]{b}$, and $\sqrt[4]{c}$.

9. $\sqrt[3]{3a}$, $\sqrt[4]{2b}$, and $\sqrt[6]{6c}$.

10. 3, $\sqrt{6}$, and $\sqrt[4]{11}$.

11. a^3, $\sqrt[m]{b^2}$, and $\sqrt[n]{c^4}$.

12. $\sqrt{a+b}$ and $\sqrt[3]{a-b}$.

13. $\sqrt[n]{2a}$, $\sqrt[2n]{2a}$, and $\sqrt[3n]{2a}$.

14. $3\sqrt{5ax^{-2}}$, $2\sqrt[3]{x^2y}$, and $4\sqrt[6]{12x^3}$.

15. $5\sqrt[6]{10x^3}$, $7\sqrt[3]{4x^2y}$, and $6\sqrt[4]{7x^3y^2}$.

Determine which is greater,

16. $\sqrt{7}$ or $\sqrt[3]{18}$.

17. $\sqrt[3]{2}$ or $\sqrt[4]{3}$.

18. $\sqrt{3}$ or $\sqrt[5]{16}$.

19. $\sqrt[3]{5}$ or $\sqrt[5]{14}$.

20. $\sqrt[4]{x^3}$ or $\sqrt[5]{x^4}$.

21. $\sqrt[4]{x^3}$ or $\sqrt[6]{x^5}$.

195. While the value of a numerical surd cannot be determined exactly, it may be determined to any required degree of accuracy by the use of decimal places in the root. To find the approximate numerical value of $\sqrt{7} \div \sqrt{5}$ we may proceed thus:

$$\frac{\sqrt{7}}{\sqrt{5}} = \frac{2.645751\cdots}{2.236068\cdots} = 1.183215\cdots.$$

This, however, involves three tedious operations, viz., extracting the square root of 7, extracting the square root of 5, and dividing the first result by the second. By multiplying numerator and denominator by $\sqrt{5}$, the second of these operations may be avoided, and the third replaced by a much shorter one. Thus,

$$\frac{\sqrt{7}}{\sqrt{5}} = \frac{\sqrt{7}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{35}}{5} = \frac{5.916079\cdots}{5} = 1.183215\cdots.$$

This suggests as suitable form for all surd fractions equivalent fractions with rational denominators (Arts. 178 and 180).

196. **Conjugate Surds** are quadratic surds which differ only in the sign of one of their terms.

Thus, $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$, $a+\sqrt{b}$ and $a-\sqrt{b}$, $\sqrt{a}+b$ and $\sqrt{a}-b$, $\sqrt{a}+\sqrt{b}+\sqrt{c}$ and $\sqrt{a}+\sqrt{b}-\sqrt{c}$, etc., are general forms of conjugate surds.

197. **Prob.** *To rationalize the surd denominator of a fraction.*

RULE. *1st. When it contains a monomial quadratic surd, multiply both terms of the fraction by this surd.*

2d. When it contains a monomial surd of any other degree, as $\sqrt[n]{x^m}$, *in which* $m < n$, *multiply both terms of the fraction by* $\sqrt[n]{x^{n-m}}$.

3d. When it contains a binomial quadratic surd, multiply both terms of the fraction by the conjugate surd.

4th. When it contains a trinomial quadratic surd, multiply both terms of the fraction by one of the surds conjugate to this, and then by the surd conjugate to the reduced product of these two surds.

DEM. 1st. $\sqrt{a}\times\sqrt{a}=a$, a rational quantity.

2d. $\sqrt[n]{x^m}\times\sqrt[n]{x^{n-m}}=\sqrt[n]{x^n}=x$, a rational quantity.

3d. $(\sqrt{a}\pm\sqrt{b})(\sqrt{a}\mp\sqrt{b})=a-b$, a rational quantity.

Also $(a\pm\sqrt{b})(a\mp\sqrt{b})=a^2-b$, a rational quantity.

4th. $(\sqrt{a}\pm\sqrt{b}\pm\sqrt{c})(\sqrt{a}\pm\sqrt{b}\mp\sqrt{c})=(\sqrt{a}\pm\sqrt{b})^2-c$
$=a+b-c\pm 2\sqrt{ab}$; and $(a+b-c\pm 2\sqrt{ab})(a+b-c\mp 2\sqrt{ab})$
$=(a+b-c)^2-4\,ab$, a rational quantity.

EXAMPLES LVIII

Reduce the following to equivalent forms having rational denominators:

1. $\dfrac{3\sqrt{5}}{4\sqrt{7\,x}}$.

OPERATION. $\dfrac{3\sqrt{5}}{4\sqrt{7\,x}}=\dfrac{3\sqrt{5}}{4\sqrt{7\,x}}\times\dfrac{\sqrt{7\,x}}{\sqrt{7\,x}}=\dfrac{3\sqrt{35\,x}}{28\,x}$, or $\dfrac{3}{28\,x}\sqrt{35\,x}$.

2. $\dfrac{6x}{5\sqrt[3]{3x^2}}$.

OPERATION. $\dfrac{6x}{5\sqrt[3]{3x^2}} = \dfrac{6x}{5\sqrt[3]{3x^2}} \times \dfrac{\sqrt[3]{3^2x}}{\sqrt[3]{3^2x}} = \dfrac{6x\sqrt[3]{9x}}{15x} = \dfrac{2\sqrt[3]{9x}}{5}$, or $\frac{2}{5}\sqrt[3]{9x}$.

3. $\dfrac{4}{\sqrt{3}}$. **4.** $\dfrac{1}{\sqrt{2}}$. **5.** $\dfrac{7x}{2\sqrt{5x}}$.

6. $\dfrac{3\sqrt{2}}{2\sqrt{3}}$. **7.** $\dfrac{6\sqrt{6x}}{7\sqrt{7x}}$. **8.** $\dfrac{8}{\sqrt{x^2-2}}$.

9. $\dfrac{2\sqrt{x^2+xy+y^2}}{3\sqrt{x-y}}$. **10.** $\dfrac{a\sqrt{x^2-xy+y^2}}{b\sqrt{x+y}}$. **11.** $3\sqrt{\frac{3}{5}}$.

12. $14\sqrt{\frac{3}{7}}$. **13.** $20x\sqrt{\dfrac{11}{5x}}$. **14.** $\dfrac{5}{\sqrt[3]{3}}$.

15. $8\sqrt[3]{\frac{4}{3}}$. **16.** $\dfrac{8a\sqrt[3]{4x}}{7b\sqrt[3]{2x^2}}$. **17.** $\sqrt[4]{\frac{2}{3}}$.

18. $\dfrac{3\sqrt[4]{3x}}{2\sqrt[4]{2x^3}}$. **19.** $7\sqrt[5]{\frac{3}{2}}$. **20.** $12\sqrt[5]{\frac{3}{4}}$.

21. $\dfrac{3ab}{\sqrt[5]{27a^4}}$. SUG. Write in the form $\dfrac{3ab}{\sqrt[5]{3^3a^4}}$.

22. $18\sqrt[5]{\dfrac{9x}{16y^4}}$. **23.** $\dfrac{16x}{\sqrt[4]{64x^3}}$. **24.** $\dfrac{1-x^2}{\sqrt[3]{169x^2}}$.

25. $\dfrac{4-x^2}{\sqrt[5]{(2-x)^2}}$. **26.** $\dfrac{x^3+y^3}{\sqrt[4]{(x+y)^3}}$. **27.** $\dfrac{x^4-y^4}{\sqrt[5]{(x^2+y^2)^3}}$.

28. $\dfrac{5}{\sqrt{5}-\sqrt{2}}$.

OPERATION. $\dfrac{5}{\sqrt{5}\quad\sqrt{2}} = \dfrac{5}{\sqrt{5}-\sqrt{2}} \times \dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}} = \dfrac{5(\sqrt{5}+\sqrt{2})}{5-3} = \frac{5}{2}(\sqrt{5}+\sqrt{2})$.

29. $\dfrac{7}{3-\sqrt{2}}$. **30.** $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$.

31. $\dfrac{5+\sqrt{3}}{5-\sqrt{3}}$. **32.** $\dfrac{5}{\sqrt{1+x^2}-x}$.

33. $\dfrac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}+x}$. **34.** $\dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}$.

35. $\dfrac{\sqrt{x^2+x}+\sqrt{x^2-x}}{\sqrt{x^2+x}-\sqrt{x^2-x}}$.

36. $\dfrac{x^2+a+x\sqrt{x^2+a}}{x+\sqrt{x^2+a}}$.

37. $\dfrac{x+1+\sqrt{x^2-1}}{x+1-\sqrt{x^2-1}}$.

38. $\dfrac{a+bx-\sqrt{a^2+b^2x^2}}{a-bx+\sqrt{a^2+b^2x^2}}$.

39. $\dfrac{12}{\sqrt{5}+\sqrt{3}-\sqrt{2}}$.

Operation. $\dfrac{12}{\sqrt{5}+\sqrt{3}-\sqrt{2}} \times \dfrac{\sqrt{5}+\sqrt{3}+\sqrt{2}}{\sqrt{5}+\sqrt{3}+\sqrt{2}} = \dfrac{6(\sqrt{5}+\sqrt{3}+\sqrt{2})}{3+\sqrt{15}}$

and

$$\frac{6(\sqrt{5}+\sqrt{3}+\sqrt{2})}{3+\sqrt{15}} \times \frac{3-\sqrt{15}}{3-\sqrt{15}} = \frac{6(-2\sqrt{3}+3\sqrt{2}-\sqrt{30})}{9-15} = \sqrt{30}+2\sqrt{3}-3\sqrt{2}.$$

40. $\dfrac{8}{\sqrt{3}+\sqrt{2}+1}$.

41. $\dfrac{1}{2\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

198. Prob. *To find the factor which will rationalize any given binomial surd.*

Solution. The general form of such surd being $\sqrt[m]{a^r} \pm \sqrt[n]{b^s}$, we have,

$$\sqrt[m]{a^r} \pm \sqrt[n]{b^s} = a^{\frac{r}{m}} \pm b^{\frac{s}{n}} = a^{\frac{nr}{mn}} \pm b^{\frac{ms}{mn}} = (a^{nr})^{\frac{1}{mn}} \pm (b^{ms})^{\frac{1}{mn}}.$$

Now, with the lower sign in all cases and with the upper sign when mn is even, $a^{nr}-b^{ms}$, which is rational, is divisible by $(a^{nr})^{\frac{1}{mn}} \pm (b^{ms})^{\frac{1}{mn}}$ (Art. 99). When mn is odd, $a^{nr}+b^{ms}$, which is rational, is divisible by $(a^{nr})^{\frac{1}{mn}} + (b^{ms})^{\frac{1}{mn}}$ (Art. 99). In either case the quotient, which can be written by forms (1) and (2), Art. 99, will be the rationalizing factor.

EXAMPLES LIX

Find the rationalizing factor of each of the following:

1. $\sqrt{3}+\sqrt[3]{5}$.

Operation. $\sqrt{3}+\sqrt[3]{5} = 3^{\frac{1}{2}}+5^{\frac{1}{3}} = 3^{\frac{3}{6}}+5^{\frac{2}{6}} = (3^3)^{\frac{1}{6}}+(5^2)^{\frac{1}{6}}$.

Now 3^3-5^2 is the difference of the same even powers (the 6th powers) of the terms of $(3^3)^{\frac{1}{6}}+(5^2)^{\frac{1}{6}}$, or $3^{\frac{1}{2}}+5^{\frac{1}{3}}$, and hence is divisible by $3^{\frac{1}{2}}+5^{\frac{1}{3}}$ (Art. 99). The quotient, which may be written by form (2), Art. 99, is the

factor by which $3^{\frac{1}{2}} + 5^{\frac{1}{3}}$, or $\sqrt{3} + \sqrt[3]{5}$, must be multiplied to produce $3^3 - 5^2$, or 2, a rational quantity.

$$(3^3 - 5^2) \div (3^{\frac{1}{2}} + 5^{\frac{1}{3}})$$
$$= (3^{\frac{1}{2}})^5 - (3^{\frac{1}{2}})^4(5^{\frac{1}{3}}) + (3^{\frac{1}{2}})^3(5^{\frac{1}{3}})^2 - (3^{\frac{1}{2}})^2(5^{\frac{1}{3}})^3 + (3^{\frac{1}{2}})(5^{\frac{1}{3}})^4 - (5^{\frac{1}{3}})^5$$
$$= \sqrt{3^5} - 9\sqrt[3]{5} + \sqrt{3^3}\,\sqrt[3]{5^2} - 15 + \sqrt{3}\,\sqrt[3]{5^4} - \sqrt[3]{5^5}.$$

2. $\sqrt[3]{3} - \sqrt[3]{2}$.

3. $2 + \sqrt[3]{4}$.

4. $\sqrt{2} - \sqrt[6]{5}$.

5. $\sqrt[3]{x^2} - \sqrt[6]{y^5}$.

ADDITION AND SUBTRACTION OF SURDS

199. Prob. *To add or subtract surds.*

RULE. *Reduce each surd to its simplest form; then to the algebraic sum of the coefficients of the similar surds annex the common surd part, and indicate the addition or subtraction of the dissimilar surds.*

This is but an application of the principles of Arts. 29 and 40.

EXAMPLES LX

Perform the following indicated combinations:

1. $\sqrt{27} + 3\sqrt{108} + 2\sqrt{75}$.

OPERATION.

$$\begin{aligned} \sqrt{27} &= \sqrt{9 \times 3} = 3\sqrt{3} \\ 3\sqrt{108} &= 3\sqrt{36 \times 3} = 18\sqrt{3} \\ 2\sqrt{75} &= 2\sqrt{25 \times 3} = 10\sqrt{3} \\ & \textit{Ans.},\ 31\sqrt{3} \end{aligned}$$

2. $3\sqrt{50} + 4\sqrt{72}$.

3. $2\sqrt{75} + 3\sqrt{245}$.

4. $5\sqrt{147} + 4\sqrt{108}$.

5. $6\sqrt{175} + 7\sqrt{252}$.

6. $4\sqrt{150} - 3\sqrt{96}$.

7. $5\sqrt{396} - \sqrt{176}$.

8. $\sqrt{108} + 3\sqrt{27} - 2\sqrt{48}$.

9. $\frac{2}{3}\sqrt{180} + \frac{1}{2}\sqrt{80} - \frac{3}{5}\sqrt{125}$.

10. $\sqrt{54} + 3\sqrt[3]{128}$.

11. $5\sqrt[3]{108} - 3\sqrt[3]{32}$.

12. $3b\sqrt[3]{128a^4b^2} - a\sqrt[3]{54ab^5}$.

13. $3x\sqrt[4]{512x} + 5\sqrt[4]{162x^5}$.

14. $5\sqrt[3]{48x^5} + x\sqrt[3]{384x^2} - 4\sqrt[3]{162x^5}$.

15. $7a\sqrt{288a^3} - 7a^2\sqrt[6]{512a^3}$.

16. $3\sqrt[5]{486x^2} + 2\sqrt[5]{64x^7}$.

17. $13\sqrt[4]{324x^4} - 2\sqrt{338x^4}$.

18. $2\sqrt{\frac{8}{3}} + 3\sqrt{\frac{3}{2}} + \sqrt{\frac{27}{8}}$.

OPERATION.
$$2\sqrt{\tfrac{8}{3}} = 4\sqrt{\tfrac{2}{3}} = 4\sqrt{\tfrac{6}{9}} = \tfrac{4}{3}\sqrt{6}$$
$$3\sqrt{\tfrac{3}{2}} = 3\sqrt{\tfrac{6}{4}} = \tfrac{3}{2}\sqrt{6}$$
$$\sqrt{\tfrac{27}{8}} = \tfrac{3}{2}\sqrt{\tfrac{3}{2}} = \tfrac{3}{2}\sqrt{\tfrac{6}{4}} = \tfrac{3}{4}\sqrt{6}$$
Ans., $\frac{43}{4}\sqrt{6}$

19. $6\sqrt{\frac{2}{3}} + 7\sqrt{24}$.

20. $2\sqrt{\frac{3}{5}} + \frac{2}{5}\sqrt{240}$.

21. $\sqrt{\dfrac{a}{4}} + \sqrt[4]{\dfrac{a^2}{81}} - \sqrt{\dfrac{a}{16}}$.

22. $6\sqrt{\frac{2}{3}} - 12\sqrt{\frac{1}{6}}$.

23. $\sqrt{\frac{3}{4}} + \sqrt{\frac{1}{3}} + \frac{1}{6}\sqrt[4]{9}$.

24. $\sqrt[6]{\frac{27}{64}} + \sqrt[4]{\frac{1}{9}} + \frac{1}{6}\sqrt[8]{81}$.

25. $\sqrt{\frac{8}{5}} + \sqrt{\frac{9}{10}} - \sqrt{\frac{5}{8}}$.

26. $\sqrt{\frac{7}{2}} + \sqrt{\frac{18}{7}} - \sqrt{\frac{25}{14}}$.

27. $3\sqrt{147} - \frac{7}{3}\sqrt{\frac{1}{3}} - 2\sqrt{\frac{1}{27}}$.

28. $18\sqrt[3]{\frac{1}{24}} + \sqrt[3]{243} - 6\sqrt[3]{\frac{1}{3}}$.

29. $7b\sqrt{18a^4 - 27a^2b^2} - 2a\sqrt{8a^2b^2 - 12b^4}$.

30. $b\sqrt[3]{8a^3b + 16a^4} + a\sqrt[3]{27b^4 + 54ab^3}$.

31. $\dfrac{a}{a + \sqrt{a^2 - x^2}} + \dfrac{a}{a - \sqrt{a^2 - x^2}}$.

32. $\dfrac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}} + \dfrac{\sqrt{a+x} - \sqrt{a-x}}{\sqrt{a+x} + \sqrt{a-x}}$.

33. $\left(\dfrac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}}\right)^2 + \left(\dfrac{\sqrt{3} - \sqrt{2}}{\sqrt{3} + \sqrt{2}}\right)^2 - \dfrac{2}{5 - 2\sqrt{6}}$.

MULTIPLICATION OF SURDS

200. Prob. *To multiply surds.*

RULE. *If the surds are of different degree, reduce them to equivalent surds of the same degree; then to the product of the coefficients annex the common root of the product of the quantities under the radical sign, and reduce the result to its simplest form.*

This is but an application of the principles of Arts. 193 and 148.

201. SCH. 1. If the quantities under the radical sign are integral, the surds should first be simplified if not already in the simplest form, as the factors that may be removed are much more readily detected before the multiplication than after. Thus, we may write,

$$\sqrt{28} \times \sqrt{72} = \sqrt{2016} = \sqrt{144 \times 14} = 12\sqrt{14};$$

but after multiplying, considerable inspection is required to find the largest square factor, and the difficulty is greater with larger numbers. We should proceed thus:

$$\sqrt{28} \times \sqrt{72} = 2\sqrt{7} \times 6\sqrt{2} = 12\sqrt{14}.$$

202. SCH. 2. When the quantities under the radical sign are fractions, it is usually best not to simplify, but to employ cancellation as far as possible. Thus,

$$\sqrt{\tfrac{3}{5}} \times \sqrt{\tfrac{2}{3}} \times \sqrt{\tfrac{15}{2}} = \sqrt{\tfrac{3}{5} \times \tfrac{2}{3} \times \tfrac{15}{2}} = \sqrt{3}.$$

203. SCH. 3. Much work is often saved by resolving the quantities under the radical sign into such factors as will show those that are common, even when the surds are in the simplest form, as this reveals any factors that may be removed after the multiplication. Thus, we may write,

$$\sqrt{42} \times \sqrt{77} = \sqrt{3234} = \sqrt{49 \times 66} = 7\sqrt{66};$$

but much inspection is saved by writing

$$\sqrt{42} \times \sqrt{77} = \sqrt{7 \times 6} \times \sqrt{7 \times 11} = \sqrt{7^2 \times 66} = 7\sqrt{66}.$$

Again,

$$\sqrt{105} \times \sqrt{70} = \sqrt{7 \times 5 \times 3} \times \sqrt{7 \times 5 \times 2} = \sqrt{7^2 \times 5^2 \times 6} = 35\sqrt{6}.$$

204. Sch. 4. If multiplicand or multiplier or each of them is a polynomial, the sum of the partial products must be taken, as in the multiplication of rational polynomials.

EXAMPLES LXI

Perform the following multiplications:

1. $\sqrt{3}$ by $\sqrt{12}$.

2. $\sqrt{21}$ by $\sqrt{7}$.

3. $\sqrt{20}$ by $\sqrt{5}$.

4. $\sqrt[3]{9}$ by $\sqrt[3]{3}$.

5. $\sqrt[3]{36\,x^2}$ by $\sqrt[3]{6\,x}$.

6. $3\sqrt{15}$ by $\sqrt{50}$.

7. $4\sqrt{6}$ by $3\sqrt{12}$.

8. $\sqrt[3]{xy^2}$ by $\sqrt[3]{x^2y}$.

9. $4\sqrt[3]{36}$ by $\sqrt[3]{48}$.

10. $2\sqrt{14}$ by $\sqrt{21}$.

11. $\sqrt{60}$ by $\sqrt{30}$.

12. $\sqrt{30}$ by $\sqrt{35}$.

13. $7\sqrt{35}$ by $\sqrt{65}$.

14. $\sqrt{39}$ by $\sqrt{91}$.

15. $\sqrt{231}$ by $\sqrt{154}$.

16. $\sqrt{91}$ by $\sqrt{182}$.

17. $\sqrt{\frac{1}{3}}$ by $\sqrt{\frac{3}{5}}$.

18. $\sqrt{\frac{3}{2}}$ by $\sqrt{\frac{5}{3}}$.

19. $5\sqrt[3]{24\,a^2b}$ by $7\,b\sqrt{32\,a}$.

Solution. As the product of the same roots equals the root of the product (Art. 148), if these surds are first reduced to the same degree, the multiplication can be performed.

Now, $$5\sqrt[3]{24\,a^2b} = 5\sqrt[3]{8 \times 3\,a^2b} = 10\sqrt[3]{3\,a^2b} = 10\sqrt[6]{9\,a^4b^2}$$
$$7\,b\sqrt{32\,a} = 7\,b\sqrt{16 \times 2\,a} = 28\,b\sqrt{2\,a} = 28\,b\sqrt[6]{8\,a^3}$$

Product, $280\,ab\sqrt[6]{72\,ab^2}$

20. $4\sqrt[3]{2}$ by $5\sqrt{3}$.

21. $2\sqrt[3]{24}$ by $3\sqrt[6]{3}$.

22. $\sqrt[3]{2\,a^2b}$ by $\sqrt[6]{16\,a^5b}$.

23. $\sqrt{12}$ by $\sqrt[10]{3}$.

24. $\sqrt[3]{3\,x}$ by $\sqrt[4]{27\,x}$.

25. $\sqrt[3]{7} \times \sqrt{2} \times \sqrt[6]{5}$.

26. $\sqrt[3]{\frac{1}{2}} \times \sqrt{\frac{2}{3}} \times \sqrt[6]{\frac{27}{2}}$.

Operation. $$\sqrt[3]{\frac{1}{2}} \times \sqrt{\frac{2}{3}} \times \sqrt[6]{\frac{27}{2}} = \sqrt[6]{\frac{1}{2^2}} \times \sqrt[6]{\frac{2^3}{3^3}} \times \sqrt[6]{\frac{3^3}{2}} = 1.$$

27. $\sqrt{\frac{5}{8}}$ by $3\sqrt[4]{\frac{2}{5}}$. **28.** $6\sqrt{\frac{1}{2}}$ by $\sqrt[3]{2}$.

29. $\sqrt{2}\times\sqrt[3]{3}\times\sqrt[4]{\frac{1}{2}}\times\sqrt[3]{\frac{1}{3}}$. **30.** $\sqrt{\frac{2}{3}}\times\sqrt[3]{\frac{1}{2}}\times\sqrt[4]{\frac{9}{2}}$.

31. $\sqrt{1+x}$ by $\sqrt[3]{1+x}$. **32.** $\sqrt{a-b}\times\sqrt{a+b}\times\sqrt[6]{a^2-b^2}$.

33. $3\sqrt{6}-4\sqrt{3}+3$ by $2\sqrt{6}+5\sqrt{3}$.

Operation.

$$\begin{array}{l}
3\sqrt{6}-4\sqrt{3}+3 \\
2\sqrt{6}+5\sqrt{3} \\
\hline
36-24\sqrt{2}+6\sqrt{6} \\
-60+45\sqrt{2} \qquad\qquad +15\sqrt{3} \\
\hline
24+21\sqrt{2}+6\sqrt{6}+15\sqrt{3}
\end{array}$$

34. $5-2\sqrt{3}$ by $4+3\sqrt{3}$.

35. $2\sqrt{x}+3\sqrt{2}$ by $6\sqrt{x}-\sqrt{2}$.

36. $\sqrt{5}+\sqrt{3}-\sqrt{2}$ by $\sqrt{5}+\sqrt{2}$.

37. $5+3\sqrt{2}$ by $5-3\sqrt{2}$.

Sug. In multiplying the sum of two quantities by the difference, as in the 37th, always apply the principle of Art. 63.

38. $3\sqrt{5}+2\sqrt{3}$ by $3\sqrt{5}-2\sqrt{3}$.

39. $2\sqrt{14}-5$ by $2\sqrt{14}+5$.

40. $7a^{\frac{3}{2}}-4a\sqrt{3a}$ by $7a^{\frac{3}{2}}+4a\sqrt{3a}$.

41. $\sqrt{5}-\sqrt{3}-\sqrt{2}$ by $\sqrt{5}+\sqrt{3}-\sqrt{2}$.

42. $\sqrt{6}+\sqrt{3}+\sqrt{5}$ by $\sqrt{5}+\sqrt{3}-\sqrt{6}$.

43. $5\sqrt{8}+6\sqrt{12}-2\sqrt{20}$ by $7\sqrt{2}-3\sqrt{3}+4\sqrt{5}$.

44. $3a-3\sqrt{ab}+2b$ by $3a+3\sqrt{ab}+2b$.

Sug. In solving examples like the last four, the terms should be so grouped as to give the product of the sum and difference of two quantities.

45. $3\sqrt{a}+\sqrt{x-9a}$ by $3\sqrt{a}-\sqrt{x-9a}$.

46. $\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2}$ by $\sqrt[3]{x}+\sqrt[3]{y}$.

47. $\sqrt[3]{5}-2\sqrt[3]{6}$ by $3\sqrt[3]{4}-\sqrt[3]{36}$.

DIVISION OF SURDS

205. Prob. *To divide surds.*

Rule. *If the surds are of different degree, reduce them to equivalent surds of the same degree; then to the quotient of the coefficients annex the common root of the quotient of the quantities under the radical sign, and reduce the result to its simplest form.*

This is but an application of the principles of Arts. 193 and 150.

206. Sch. When the division gives rise to a fraction with a surd in the denominator, this denominator should be rationalized.

EXAMPLES LXII

Perform the following divisions:

1. $18\sqrt{6}$ by $3\sqrt{10}$.

Operation. $\dfrac{18\sqrt{6}}{3\sqrt{10}}=6\sqrt{\dfrac{6}{10}}=6\sqrt{\dfrac{3}{5}}=\dfrac{6}{5}\sqrt{15}.$

2. $8a\sqrt{6a}$ by $4\sqrt[3]{2a^2}$.

Operation. $\dfrac{8a\sqrt{6a}}{4\sqrt[3]{2a^2}}=2a\dfrac{\sqrt[6]{216a^3}}{\sqrt[6]{4a^4}}=2a\sqrt[6]{\dfrac{54}{a}}=2\sqrt[6]{54a^5}.$

3. $6\sqrt{12}$ by $3\sqrt{3}$.

4. $10\sqrt{15}$ by $2\sqrt{5}$.

5. $\sqrt{32}$ by $\sqrt{8}$.

6. $\sqrt{42}$ by $\sqrt{84}$.

7. $\sqrt[3]{80a^2}$ by $\sqrt[3]{5a}$.

8. $a^2\sqrt[5]{192}$ by $a\sqrt[5]{3}$.

9. $\sqrt{x^3+y^3}$ by $\sqrt{x+y}$.

10. $\sqrt[4]{(x^2-3y)^5}$ by $\sqrt{(x^2-3y)^3}$.

11. $\sqrt{30}$ by $\sqrt{42}$.

12. $\sqrt{15}$ by $\sqrt[3]{15}$.

13. 6 by $\sqrt[3]{6}$.

14. $\sqrt[3]{81a^2}$ by $3a$.

15. $\sqrt{10}$ by $\sqrt[6]{40}$.

16. $\sqrt{2}$ by $\sqrt[10]{8}$.

17. $\sqrt[4]{x^3}$ by $\sqrt[3]{x^2}$.

18. $a^2b\sqrt{12ab}$ by $ab^2\sqrt[3]{24ab}$.

19. $\sqrt{\frac{5}{7}}$ by $\sqrt{\frac{10}{21}}$.

20. $\sqrt{\frac{21}{33}}$ by $\sqrt{\frac{28}{44}}$.

21. $\sqrt[12]{\frac{27}{5}}$ by $\sqrt[4]{\frac{3}{5}}$.

22. $5-\sqrt{135}$ by $\sqrt{5}$.

23. $6\sqrt{32}-12\sqrt{24}+3\sqrt{5}$ by $3\sqrt{2}$.

24. $\sqrt{14}-\sqrt{35}$ by $\sqrt{2}-\sqrt{5}$.

25. $3-\sqrt[3]{3}+\sqrt[4]{3}$ by $\sqrt{3}$.

26. $2\sqrt{3}+7\sqrt{2}$ by $5\sqrt{3}-4\sqrt{2}$.

Sug. Write as a fraction and rationalize the denominator.

27. $5\sqrt{5}+3\sqrt{2}$ by $\sqrt{5}+\sqrt{2}$.

28. $a+b-c+2\sqrt{ab}$ by $\sqrt{a}+\sqrt{b}-\sqrt{c}$.

29. $\dfrac{x^2+\sqrt{x^4-a^4}}{x^2-\sqrt{x^4-a^4}}-\dfrac{x^2-\sqrt{x^4-a^4}}{x^2+\sqrt{x^4-a^4}}$ by $4\sqrt{\dfrac{x^2-a^2}{x^2+a^2}}$.

30. $\sqrt{5+2\sqrt{6}}-\sqrt{5-2\sqrt{6}}$ by $\sqrt{5+2\sqrt{6}}+\sqrt{5-2\sqrt{6}}$.

INVOLUTION OF SURDS

207. Prob. *To raise a monomial surd to any power.*

Rule. *Either divide the index of the root by the exponent of the power, or raise the quantity under the radical sign to the required power; then annex the result to the required power of the coefficient, and reduce to the simplest form.*

This is but an application of the principles of Arts. 148 and 151.

208. Cor. *A surd is raised to a power whose exponent is the index of the root by simply removing the radical sign.*

EXAMPLES LXIII

Perform the following indicated operations:

1. $(5\sqrt{3x})^2$. **2.** $(-6\sqrt{\tfrac{2}{3}x})^2$. **3.** $(4\sqrt[3]{5x^2y})^2$.

4. $(3\sqrt[3]{-2ab^2})^3$. **5.** $(-\tfrac{2}{5}\sqrt{5a})^3$. **6.** $(3\sqrt[4]{14a})^2$.

7. $(\sqrt[6]{53a^2b^3})^3$. **8.** $(2\sqrt[12]{8a^5b})^4$. **9.** $(\sqrt[5]{2a^3b^2})^6$.

10. $(\sqrt[9]{7a^3b^5})^3$. **11.** $(\sqrt[8]{50xy})^4$. **12.** $\left(\sqrt[3]{\sqrt[5]{-27a^3b^6}}\right)^5$.

13. $\left(\sqrt[3]{\sqrt[6]{64x^6y^2}}\right)^3$. **14.** $(\sqrt{7}-\sqrt{5})^2$. **15.** $(3+\sqrt{6})^2$.

16. $(\sqrt{3}-\sqrt{6})^2$. **17.** $(\sqrt[3]{2}-\sqrt[3]{3})^3$. **18.** $(\sqrt[6]{3}+\sqrt[3]{2})^3$.

EVOLUTION OF SURDS

209. Prob. *To extract any root of a monomial surd.*

Rule. *Either extract the required root of the quantity under the radical sign, or multiply the index of the given root by the index of the required root, then annex the result to the required root of the coefficient, and reduce to the simplest form.*

This is but an application of the principles of Arts. 148 and 151.

EXAMPLES LXIV

Perform the following indicated operations:

1. $\sqrt{(\sqrt[3]{a^2b^2})}$.

2. $\sqrt{(36\sqrt{25\,a^4b^2})}$.

3. $\sqrt[3]{(-27\sqrt[3]{64\,a^3b^6})}$.

4. $\sqrt[5]{(a^{10}b^5\sqrt[4]{32\,a^5b^{10}})}$.

5. $\sqrt[3]{(8\sqrt{7\,ax})}$.

6. $\sqrt{(\sqrt[5]{1024\,x^2y^4})}$.

7. $\sqrt[4]{a\sqrt[3]{a}}$.

8. $\sqrt[5]{8\,a^5\sqrt[3]{2}}$.

9. $\sqrt[4]{(43\,xy^2\sqrt[3]{43\,xy^2})}$.

10. $\sqrt[5]{x^5(x-y)^{5n}}$.

11. $\sqrt{(\sqrt[3]{49-70\,x^2+25\,x^4})}$.

12. $\sqrt[7]{(64\,a^3b^3\sqrt[6]{64\,a^3b^3})}$.

210. Theorem. *The square root of a binomial, one of whose terms is rational and the other a quadratic surd, can be found whenever the rational term is separable into two parts the product of whose square roots is half of the surd term.*

Dem. Let $a \pm b\sqrt{c}$ be such that $a = m + n$, and $\pm\frac{1}{2}b\sqrt{c} = \sqrt{mn}$. Then $a \pm b\sqrt{c} = m + n \pm 2\sqrt{mn} = (\sqrt{m} \pm \sqrt{n})^2$. Therefore the square root is $\sqrt{m} \pm \sqrt{n}$.

EXAMPLES LXV

Extract the square root of each of the following:

1. $27 - 10\sqrt{2}$.

Solution. $27 - 10\sqrt{2} = 25 - 10\sqrt{2} + 2 = (5 - \sqrt{2})^2$. Hence the square root is $5 - \sqrt{2}$. Of course, the negative of this root, viz., $\sqrt{2} - 5$, is equally admissible.

2. $12 - 2\sqrt{35}$.

SOLUTION. $12 - 2\sqrt{35} = 7 - 2\sqrt{35} + 5 = (\sqrt{7} - \sqrt{5})^2$. Hence the square root is $\sqrt{7} - \sqrt{5}$.

3. $4 + 2\sqrt{3}$.
4. $3 + 2\sqrt{2}$.
5. $7 - 2\sqrt{10}$.
6. $8 - 4\sqrt{3}$.
7. $30 + 10\sqrt{5}$.
8. $8 - 2\sqrt{15}$.
9. $70 - 30\sqrt{5}$.

SOLUTION. $70 - 30\sqrt{5} = 25 - 30\sqrt{5} + 9 \times 5 = (5 - 3\sqrt{5})^2$. Hence the square root is $5 - 3\sqrt{5}$.

10. $18 + 8\sqrt{5}$.
11. $11 - 4\sqrt{6}$.
12. $49 + 12\sqrt{5}$.

211. NOTE. The method of inspection here given is of limited application. A general method, applicable to all perfect squares of the kind mentioned in the theorem, requiring, however, the solution of two equations, one of the first and the other of the second degree, will be given as an application of the process of solving such equations.

SECTION II—IMAGINARY QUANTITIES

212. **An Imaginary Quantity** is an expression which contains an indicated even root of a negative quantity.

Thus, $\sqrt{-1}$, $\sqrt{-x^2}$, $\sqrt[6]{-12}$, $3 \pm 2\sqrt[4]{-4}$, $a \pm b\sqrt{-1}$ are imaginary quantities.

213. All quantities not imaginary are called **Real Quantities.**

214. Expressions which contain both real and imaginary terms are often called **Complex Quantities.**

215. When the radical sign affects a negative quantity, we cannot regard it as indicating a possible arithmetical operation. For example, $\sqrt{-9}$ is neither $+3$ nor -3, for neither multiplied by itself, *i.e.* squared, will produce -9, but $+9$ instead. Since imaginary quantities occur frequently in mathematical investigations and lead to important results, we need to consider what meaning

should be attached to them in order that they may obey the ordinary laws of algebra.

The expression $\sqrt{a}$ is understood to be such that $\sqrt{a}\times\sqrt{a}=a$. Now if we agree that $\sqrt{-a}$ is such that, in the same way, $\sqrt{-a}\times\sqrt{-a}=-a$, we shall find that imaginary quantities obey the algebraic laws already established. This agreement would forbid our first multiplying together the quantities under the radical sign, observing the law of signs, and then extracting the square root; for in that ease we would have $\sqrt{-a}\times\sqrt{-a}=\sqrt{a^2}=\pm a$. The agreement limits us to the minus sign, as it should, since the square root of a quantity multiplied by itself should produce the original quantity.

216. Theorem. *Every monomial imaginary can be reduced to the form $b\sqrt[n]{-1}$, in which b may be either rational or surd.*

Dem. The general form of a monomial imaginary is $m\sqrt[n]{-k}$, in which n is even. Now, since the root of the product equals the product of the roots, $m\sqrt[n]{-k}=m\sqrt[n]{k(-1)}=m\sqrt[n]{k}\sqrt[n]{-1}=b\sqrt[n]{-1}$, where $b=m\sqrt[n]{k}$.

217. Theorem. *Every polynomial containing some real terms and some imaginary terms of the same degree can be reduced to the form $a\pm b\sqrt[n]{-1}$, in which a and b may be either rational or surd.*

Dem. This is evident from the fact that all the real terms can be combined into one (it may be a polynomial) and represented by a, and the imaginary terms into another represented by $\pm b\sqrt[n]{-1}$.

218. Cor. *The general form of an imaginary of the second degree is $a\pm b\sqrt{-1}$, in which b is rational or surd, and a is rational, surd, or 0.*

219. The Imaginary Element is some even root of -1 which renders an expression imaginary.

Thus, in $\sqrt[6]{-x}=\sqrt[6]{x}\sqrt[6]{-1}$, $5\sqrt{-x^2}=5x\sqrt{-1}$,

$$2+\sqrt{-4}=2+2\sqrt{-1},\quad \sqrt[4]{-32}=2\sqrt[4]{-2}=2\sqrt[4]{2}\sqrt[4]{-1},$$

the imaginary elements are $\sqrt[6]{-1}$, $\sqrt{-1}$, $\sqrt[4]{-1}$.

$\sqrt{-1}$, sometimes called the *Imaginary Unit*, is often represented by i.

220. Theorem. *When the index of the imaginary element is a composite number and one of the factors odd, the value is not changed by rejecting the odd factor.*

DEM. In the form $b\sqrt[n]{-1}$ let the factors of n be p and q, q being odd. Then $b\sqrt[n]{-1}=b\sqrt[pq]{-1}=b\sqrt[p]{\sqrt[q]{-1}}=b\sqrt[p]{-1}$, since any odd root of -1 is -1.

Thus,
$$3\sqrt[6]{-1}=3\sqrt{\sqrt[3]{-1}}=3\sqrt{-1},$$
$$\sqrt{5}\sqrt[10]{-1}=\sqrt{5}\sqrt{\sqrt[5]{-1}}=\sqrt{5}\sqrt{-1},$$
$$2\sqrt[12]{-1}=\sqrt[4]{\sqrt[3]{-1}}=\sqrt[4]{-1},$$
$$\sqrt[6]{-8}=\sqrt[6]{8}\sqrt[6]{-1}=\sqrt{\sqrt[3]{8}}\sqrt{\sqrt[3]{-1}}=\sqrt{2}\sqrt{-1}.$$

221. Conjugate Imaginaries are imaginaries of the second degree which differ only in the sign of the imaginary part.

Thus, $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$, $5+3\sqrt{-1}$ and $5-3\sqrt{-1}$, $m\sqrt{-1}$ and $-m\sqrt{-1}$, $4\sqrt{-1}$ and $-4\sqrt{-1}$, etc., are conjugate imaginaries.

222. Theorem. *Both the sum and the product of two conjugate imaginaries are real.*

DEM. $$b\sqrt{-1}+(-b\sqrt{-1})=0,$$
and $$a+b\sqrt{-1}+a-b\sqrt{-1}=2a.$$
Also $$b\sqrt{-1}\times(-b\sqrt{-1})=-b^2(-1)=b^2,$$
and $$(a+b\sqrt{-1})\times(a-b\sqrt{-1})=a^2+b^2,$$
the last being the product of the sum and difference of two quantities, which is the difference of their squares.

223. The Modulus of a quadratic imaginary of the form $a\pm b\sqrt{-1}$ is the positive square root of a^2+b^2.

Each of the imaginaries that constitute a conjugate pair has the same modulus, and this common modulus is seen to be the positive square root of the product of the two.

224. Prob. *To add or subtract imaginaries.*

RULE. *Reduce each to the form in which the imaginary element is expressed by the smallest possible index* (*Art.* 220). *If the imagi-*

nary element is then the same in each, combine its coefficients and write the result as the coefficient of the common imaginary element; if the imaginary element is not the same in each, indicate the addition or subtraction.

If the imaginary element is the same in each, the problem is simply that of uniting quantities of the same kind.

EXAMPLES LXVI

Combine the following:

1. $6+\sqrt{-4}$ and $8+\sqrt{-9}$.

OPERATION.
$$\begin{aligned} 6+\sqrt{-4} &= 6+2\sqrt{-1} \\ 8+\sqrt{-9} &= 8+3\sqrt{-1} \\ \textit{Sum},\ & \quad 14+5\sqrt{-1} \end{aligned}$$

2. $2\sqrt[6]{-8}$ and $5\sqrt{-2}$.

OPERATION.
$$\begin{aligned} 2\sqrt[6]{-8} = 2\sqrt{\sqrt[3]{-8}} = 2\sqrt{-2} &= 2\sqrt{2}\sqrt{-1} \\ 5\sqrt{-2} &= 5\sqrt{2}\sqrt{-1} \\ \textit{Sum},\ & \quad 7\sqrt{2}\sqrt{-1} \end{aligned}$$

3. $4\sqrt{-27}$ and $6\sqrt{-32}$.

OPERATION.
$$4\sqrt{-27} = 4\sqrt{9\times 3\times(-1)} = 12\sqrt{3}\sqrt{-1},$$
$$6\sqrt{-32} = 6\sqrt{16\times 2\times(-1)} = 24\sqrt{2}\sqrt{-1}.$$

Here, although the imaginary element is the same in both quantities, they contain dissimilar surds. Hence we cannot unite into one term, but have

$$12\sqrt{3}\sqrt{-1}+24\sqrt{2}\sqrt{-1} = 12(\sqrt{3}+2\sqrt{2})\sqrt{-1}.$$

4. $5a+3b\sqrt{-25}$ and $3a+2b\sqrt{-4}$.

5. $15\sqrt{-16}$ and $6\sqrt{-9}$.

6. $\sqrt{-225}$ and $\sqrt{-169}$.

7. $\sqrt{-9x^2}$ and $\sqrt{-4x^2}$.

8. $3\sqrt{-9}$ and $-\sqrt{-49}$.

9. $5\sqrt{-12}$ and $2\sqrt{-48}$.

10. $\sqrt{-289}$ and $-\sqrt{-9}$.

11. $a\sqrt{-8}$ and $\sqrt{-18a^2}$.

12. $7+4\sqrt{-27}$ and $3-2\sqrt{-12}$.

13. $\sqrt[4]{-16}$ and $-\sqrt[4]{-1}$.

14. $4+2\sqrt[4]{-a}$ and $2-\sqrt[4]{-a}$.

15. $\sqrt[10]{-32}$ and $\sqrt[6]{-8}$.

16. $\sqrt{-144}-\sqrt[6]{-27}-\sqrt{-121}+\sqrt[10]{-243}$.

225. Prob. *To determine the different powers of $\sqrt{-1}$ and $\sqrt[4]{-1}$.*

SOLUTION.

$$(\sqrt{-1})^1=\sqrt{-1},$$
$$(\sqrt{-1})^2=-1, \text{ by Art. 215},$$
$$(\sqrt{-1})^3=(\sqrt{-1})^2\sqrt{-1}=-\sqrt{-1},$$
$$(\sqrt{-1})^4=[(\sqrt{-1})^2]^2 \quad =1,$$
$$(\sqrt{-1})^5=(\sqrt{-1})^4\sqrt{-1}=\sqrt{-1},$$
$$(\sqrt{-1})^6=(\sqrt{-1})^5\sqrt{-1}=-1,$$
$$(\sqrt{-1})^7=(\sqrt{-1})^6\sqrt{-1}=-\sqrt{-1},$$
$$(\sqrt{-1})^8=[(\sqrt{-1})^4]^2 \quad =1:$$
etc.

It is thus seen that we have a succession of the results, $\sqrt{-1}$, -1, $-\sqrt{-1}$, and 1, or that the only forms assumed by the different powers of $\sqrt{-1}$ are $\pm\sqrt{-1}$ and ∓ 1.

$$(\sqrt[4]{-1})^1=\sqrt[4]{-1},$$
$$(\sqrt[4]{-1})^2=\sqrt{-1},$$
$$(\sqrt[4]{-1})^3=(\sqrt[4]{-1})^2\sqrt[4]{-1}=\sqrt{-1}\sqrt[4]{-1},$$
$$(\sqrt[4]{-1})^4=-1,$$
$$(\sqrt[4]{-1})^5=(\sqrt[4]{-1})^4\sqrt[4]{-1}=-\sqrt[4]{-1},$$
$$(\sqrt[4]{-1})^6=(\sqrt[4]{-1})^4(\sqrt[4]{-1})^2=-\sqrt{-1},$$
$$(\sqrt[4]{-1})^7=(\sqrt[4]{-1})^6\sqrt[4]{-1}=-\sqrt{-1}\sqrt[4]{-1},$$
$$(\sqrt[4]{-1})^8=[(\sqrt[4]{-1})^4]^2=1,$$
etc.

It is thus seen that the only forms assumed by the different powers of $\sqrt[4]{-1}$ are $\pm\sqrt[4]{-1}$, $\pm\sqrt{-1}$, $\pm\sqrt{-1}\sqrt[4]{-1}$, and ∓ 1.

226. Prob. *To multiply or divide imaginaries.*

RULE. *Reduce to the form in which the root index of the imaginary element is the same in each imaginary term; and multiply or divide as in case of other quantities, observing the principles of Art. 225 for the product or quotient of the imaginary elements.*

EXAMPLES LXVII

Multiply the following:

1. $4\sqrt{-3}$ by $2\sqrt{-2}$.

OPERATION.
$$4\sqrt{-3} = 4\sqrt{3}\sqrt{-1}$$
$$2\sqrt{-2} = 2\sqrt{2}\sqrt{-1}$$
$$\textit{Product,}\quad 8\sqrt{6}(-1) = -8\sqrt{6}.$$

2. $7\sqrt[4]{-32}$ by $5\sqrt[4]{-2}$.

OPERATION.
$$7\sqrt[4]{-32} = 7\sqrt[4]{16}\sqrt[4]{-2} = 14\sqrt[4]{2}\sqrt[4]{-1}$$
$$5\sqrt[4]{-2} = 5\sqrt[4]{2}\sqrt[4]{-1}$$
$$\textit{Product,}\quad 70\sqrt{2}(\sqrt[4]{-1})^2,$$
$$\text{or}\quad 70\sqrt{2}\sqrt{-1}.$$

3. $\sqrt{-x^2}$ by $\sqrt{-y^2}$.

4. $5\sqrt{-5}$ by $4\sqrt{-3}$.

5. $-2\sqrt{-25}$ by $3\sqrt{-36}$.

6. $\sqrt{-2}$ by $\sqrt{-8}$.

7. $\sqrt{-196}$ by $\sqrt{-27}$.

8. $3\sqrt{-12}$ by $4\sqrt{-3}$.

9. $14\sqrt{-1}+\sqrt{-2}$ by $2\sqrt{-1}$.

10. $4+2\sqrt{-4}$ by $5-3\sqrt{-4}$.

11. $1+\sqrt{-1}$ by $1-\sqrt{-1}$.

12. $12+3\sqrt{-1}$ by $12-3\sqrt{-1}$.

13. $3\sqrt[4]{-9}$ by $4\sqrt[6]{-8}$.

OPERATION.
$$3\sqrt[4]{-9} = 3\sqrt[4]{9}\sqrt[4]{-1} = 3\sqrt{3}\sqrt[4]{-1}$$
$$4\sqrt[6]{-8} = 4\sqrt[6]{8}\sqrt[6]{-1} = 4\sqrt{2}\sqrt{-1} = 4\sqrt{2}(\sqrt[4]{-1})^2$$
$$\textit{Product,}\quad 12\sqrt{6}(\sqrt[4]{-1})^3 = 12\sqrt{6}\sqrt{-1}\sqrt[4]{-1}.$$

14. $3\sqrt[4]{-2}$ by $5\sqrt{-2}$.

15. $3\sqrt[4]{-2}$ by $5\sqrt[6]{-3}$.

Find the value of each of the following:

16. $(5\sqrt{-2})^2$.

17. $(2\sqrt{-3})^3$.

18. $(4\sqrt[6]{-2})^4$.

19. $(3\sqrt{-2})^5$.

20. $(2\sqrt[4]{-2})^6$.

21. $(\sqrt{-7})^7$.

Find the modulus of each of the following:

22. $\sqrt{2}+\sqrt{-2}$.

23. $\sqrt{2}-\sqrt{-2}$.

24. $3+2\sqrt{-3}$.

25. $3-2\sqrt{-3}$.

26. $4-3\sqrt{-1}$.

27. $4+3\sqrt{-1}$.

Rationalize the denominators of the following:

28. $\dfrac{1}{3-\sqrt{-2}}$.

29. $\dfrac{\sqrt{-2}+2\sqrt{-3}}{\sqrt{-2}-2\sqrt{-3}}$.

30. $\dfrac{2}{\sqrt{-9}}$.

31. $\dfrac{3\sqrt{-2}+2\sqrt{-5}}{3\sqrt{-2}-2\sqrt{-5}}$.

Divide the following:

32. $6\sqrt{-16}$ by $2\sqrt{-4}$.

OPERATION. $$\frac{6\sqrt{-16}}{2\sqrt{-4}}=\frac{24\sqrt{-1}}{4\sqrt{-1}}=6.$$

The imaginary elements cancel.

33. $2\sqrt{-1}$ by $\sqrt[4]{-2}$.

OPERATION. $$\frac{2\sqrt{-1}}{\sqrt[4]{-2}}=\frac{2(\sqrt[4]{-1})^2}{\sqrt[4]{2}\sqrt[4]{-1}}=\frac{2\sqrt[4]{-1}}{\sqrt[4]{2}}=\sqrt[4]{8}\sqrt[4]{-1}=\sqrt[4]{-8}.$$

34. $16\sqrt{-9}$ by $2\sqrt{-4}$.

35. $4\sqrt{-1}$ by $2\sqrt{-3}$.

36. 1 by $\sqrt{-1}$.

37. 36 by $12\sqrt{-1}$.

38. $63\sqrt[6]{-16}$ by $\sqrt{-81}$.

39. $\sqrt[6]{-16}$ by $\sqrt[4]{-4}$.

40. $4+\sqrt{-2}$ by $2-\sqrt{-2}$.

OPERATION. When there is more than one term in the divisor, it is usually best to write the quotient as a fraction and then rationalize the denominator. Thus,

$$\frac{4+\sqrt{-2}}{2-\sqrt{-2}}=\frac{4+\sqrt{2}\sqrt{-1}}{2-\sqrt{2}\sqrt{-1}}=\frac{(4+\sqrt{2}\sqrt{-1})(2+\sqrt{2}\sqrt{-1})}{4+2}$$
$$=\frac{6+6\sqrt{2}\sqrt{-1}}{6}=1+\sqrt{2}\sqrt{-1}.$$

41. $1+\sqrt{-1}$ by $1-\sqrt{-1}$.

42. 1 by $3-2\sqrt{-3}$.

43. $a+\sqrt{-x}$ by $a-\sqrt{-x}$.

44. $x-\dfrac{1+\sqrt{-3}}{2}$ by $\dfrac{2}{2x-1+\sqrt{-3}}$.

45. Simplify $\dfrac{a+\sqrt{-b}}{a-\sqrt{-b}}+\dfrac{a-\sqrt{-b}}{a+\sqrt{-b}}$.

Extract the square root of each of the following:

46. $5 + 12\sqrt{-1}$.

OPERATION. $5 + 12\sqrt{-1} = 9 + 12\sqrt{-1} - 4 = (3 + 2\sqrt{-1})^2$. Hence the square root is $3 + 2\sqrt{-1}$.

47. $-5 + 12\sqrt{-1}$.

SUG. Write in the form $4 + 12\sqrt{-1} - 9$.

48. $-7 - 24\sqrt{-1}$.

SUG. Write in the form $9 - 24\sqrt{-1} - 16$.

49. $7 + 12\sqrt{-4}$.

50. $21 - 20\sqrt{-1}$.

51. $-9 + 20\sqrt{-4}$.

52. $35 - 12\sqrt{-1}$.

53. $40 + 42\sqrt{-1}$.

54. $33 - 56\sqrt{-1}$.

55. $27 + 36\sqrt{-1}$.

56. $24 - 70\sqrt{-1}$.

57. Simplify $\dfrac{3 + 2\sqrt{-1}}{2 - 5\sqrt{-1}} + \dfrac{3 - 2\sqrt{-1}}{2 + 5\sqrt{-1}}$.

58. Simplify $\dfrac{\sqrt{3 - x} - \sqrt{-2}}{\sqrt{3 - x} + \sqrt{-2}} - \dfrac{\sqrt{-x} + \sqrt{-5}}{\sqrt{-x} - \sqrt{-5}}$.

227. In Algebra imaginary quantities have their greatest practical use in showing that, arithmetically, the conditions of problems in the solution of which they occur cannot be fulfilled. For example, let it be required to divide 10 into two parts whose product shall be 40. Proceeding in the usual way we find the two parts to be $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$. These imaginary results show that the requirements of the problem cannot, in the arithmetical sense, be fulfilled. It is easy to show that the largest product of any two parts of 10 is 25. Nevertheless, these imaginary values do satisfy the algebraic requirements of the problem, the sum of $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ being 10 and the product 40. Imaginary quantities occur as well, and always in conjugate pairs, as roots of an equation which also has one or more real roots.

Imaginaries also have a practical use in Algebra in showing the limits of a variable quantity; for example, in finding, by algebraic methods, the maximum or minimum value of a function.

In Analytical Geometry imaginaries have a frequent and important use in showing the limits of loci represented by certain forms of equations.

In a branch of Mathematics known as Quaternions imaginary quantities have a graphical and definite signification. If any magnitude be represented by a, then $-a$ represents an equal magnitude in the opposite direction. Now $a \times (-1) = -a$; hence -1, regarded as an operator, causes a reversal. We have seen that $a \times \sqrt{-1} \times \sqrt{-1} = -a$; hence $\sqrt{-1}$, regarded as an operator, causes a reversal when repeated.

Now any magnitude may be represented graphically by a length laid off on a straight line. Suppose this length to be a. Then $-a$ would be the same length in the opposite direction. Now we may, if we choose, regard $\sqrt{-1}$ as being an operator which turns a through a right angle, because, when repeated in the same direction, it causes a reversal. Thus, just as $a \times \sqrt{-1} \times \sqrt{-1} = -a$, a turned successively through two right angles gives $-a$; and $\sqrt{-1}$, represented by i in Quaternions, is the operator which, when repeated, causes this reversal.

CHAPTER XI

SIMPLE EQUATIONS

SECTION I — SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY

228. An Equation is an assertion by means of a mathematical symbol that two expressions have the same value. This symbol is the *Sign of Equality* (Art. 10).

229. The Members of an Equation are the expressions connected by the sign of equality, the one on the left being called the **First Member** and the one on the right the **Second Member.**

230. An Identical Equation, called also an **Identity,** is an equation that is true for all values of the letter or letters involved.*

Thus, $(x+a)(x-a)=x^2-a^2$ is an identity.

231. An Equation of Condition, usually called simply an **Equation,** is an equation that is true only for a limited number of values of the letter or letters involved.

Thus, $x+5=9$ is true only for $x=4$; and $x^2-6x=7$ is true only for $x=7$ and $x=-1$.

232. The Unknown Quantity is the letter whose value is required.

233. An Absolute Term of an equation is a term which does not contain an unknown quantity.

234. A Numerical Equation is one in which the known quantities are represented by numbers.

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* Some writers use the sign $\equiv$ to indicate that two expressions are identically equal. The sign is not used in this work, as the distinction is not thought to be of sufficient importance.

235. **A Literal Equation** is one in which some or all of the known quantities are represented by letters.

236. **The Degree of an Equation** is the same as that of its term of highest degree (Art. 50), the unknown quantity or quantities first being freed from fractional and negative exponents.

Thus, $ax^2 + bx = c$ is of the second degree, $3x^2y - 5xy + 2x + 6y^2 = 12$ is of the third degree, $xy = m$ is of the second degree, $\frac{1}{x} + \frac{1}{y} = 5$ (which, by multiplying both members by xy, becomes $y + x = 5xy$) is of the second degree, $x^2 + 3x - 5x^{-2} - 3 = 0$ (which, by multiplying both members by x^2, becomes $x^4 + 3x^3 - 5 - 3x^2 = 0$) is of the fourth degree.

237. **A Simple** or **Linear Equation** is one of the first degree.

238. Equations of one unknown quantity are called **Quadratic, Cubic, Biquadratic,** or **Quintic,** according as they are of the second, third, fourth, or fifth degree.

239. Any equation above the second degree is called a **Higher Equation.**

240. **A Root of an Equation** is a value of the unknown quantity which renders the equation true.

241. **To solve an Equation** is to find its root or roots.

242. An equation is said to be **Satisfied,** and a supposed root **Verified,** when, on substituting for the unknown quantity the supposed root, the equation becomes an identity.

243. In solving equations the following axioms are employed:

1. *If the same quantity or equal quantities be added to or subtracted from both members of an equation, the equality of the members will not be destroyed.*

2. *If both members of an equation be multiplied or divided by any quantity that is not equal to 0, the equality of the members will not be destroyed.**

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* For the introduction and loss of roots by multiplying or dividing by a factor containing the unknown quantity, see Art. 351.

To show that the equality of the members may be destroyed by dividing both members by a quantity that is equal to 0, let us take the equation

$$5x - 15 = 2x - 6, \tag{1}$$

which is seen to be satisfied for $x = 3$.

By factoring we have

$$5(x - 3) = 2(x - 3). \tag{2}$$

Now if we should divide both members of (2) by $x - 3$, we should have the absurd result $5 = 2$. The two members of (2) are equal, not because of the relation of 5 to 2, but because $x - 3 = 0$, and 5 times 0 is the same as 2 times 0. In dividing by $x - 3$ we remove the element that makes the two members equal.

244. Prob. *To solve a simple equation with one unknown quantity.*

Rule. 1. *Clear the equation of fractions, if it have any, by multiplying each term by the* **l. c. m.** *of all the denominators.*

2. *Transpose the unknown terms to the first member and the known terms to the second member, by changing the sign of each term transposed.*

3. *Unite similar terms, and divide both members by the coefficient of the unknown quantity.*

Dem. 1. Multiplying by the **l. c. m.** of all the denominators does not destroy the equality of the members (Art. 243, 2), and it clears of fractions because in multiplying any one of the fractions by this **l. c. m.** the denominator of that term is canceled by one factor of the multiplier.

2. When a term is dropped from one member, it is subtracted from that member. Now if it is written with its sign changed in the other member, it is subtracted from that member also, and the equality of the members is not destroyed (Art. 243, 1).

3. Dividing by the coefficient of the unknown quantity, after uniting terms, does not destroy the equality of the members (Art. 243, 2), and it leaves in the first member simply the unknown quantity; hence the second member is its value, or the root of the equation.

245. Cor. *The signs of all the terms of an equation may be changed without destroying the equality of the members.*

For this is the same as multiplying or dividing by -1.

246. Sch. Equations having terms of higher degree than the first often reduce to simple equations by the disappearance of these terms in collecting.

SOME PRACTICAL SUGGESTIONS

247. 1. When there are several integral terms and but few fractional terms, time is saved by collecting terms before clearing of fractions.

2. In clearing of fractions the student must be careful to change the signs of the terms in the numerator of a fraction that is preceded by the minus sign (Art. 11).

3. In simple cases the terms may be transposed and united at the same time.

4. Factors which appear in both members of the equation should be canceled, and equal terms with opposite signs in the same member of the equation, or the same sign in opposite members of the equation, should be stricken out.

5. When a fraction has a polynomial numerator and monomial denominator, it is often better to separate the fraction into parts by dividing each term of the numerator separately by the denominator.

6. It is often expedient to clear of the smaller or simpler denominators first, and after each step to see that by transposition, uniting terms, etc., the equation is kept in as simple a form as possible.

EXAMPLES LXVIII

Solve the following:

1. $3x+\frac{2x-10}{3}=\frac{3x}{2}-\frac{5x-14}{4}.$

Solution. Clearing of fractions by multiplying every term by 12, the l. c. m. of the denominators, we have

$$36x+8x-40=18x-15x+42.$$

Observe that the minus sign before the last term of the given equation denotes that the whole term is to be subtracted. Hence when the vinculum is removed in clearing of fractions, we must either still indicate this subtraction by writing $-(15x-42)$, or perform it by changing the signs.

Transposing, $36x+8x-18x+15x=42+40.$

Collecting terms, $41x=82.$

Dividing by the coefficient of x, $x=2.$

2. $\dfrac{x-a}{2}=\dfrac{(x-b)^2}{2x-a}.$

SOLUTION. Performing the indicated involution and clearing of fractions,

$$2x^2-2ax-ax+a^2=2x^2-4bx+2b^2.$$

dropping $2x^2$ from both members, transposing and uniting,

$$4bx-3ax=2b^2-a^2,$$

or

$$(4b-3a)x=2b^2-a^2.$$

$$\therefore\ x=\frac{2b^2-a^2}{4b-3a}.$$

The answer would be equally correct if written

$$x=\frac{a^2-2b^2}{3a-4b}.\quad \text{Why ?}$$

3. $8x-5(4x+3)=25-8x.$

4. $5(x-2)-6(x+4)=21.$

5. $\dfrac{x-3}{2}+\dfrac{x}{3}=20-\dfrac{x-19}{2}.$

6. $2x-\dfrac{3x+7}{11}=\dfrac{x}{2}+1.$

7. $\dfrac{3x+4}{2}-\dfrac{19x-3}{8}=\dfrac{4-7x}{12}.$

8. $\dfrac{6x-4}{3}-2=\dfrac{18-4x}{3}+x.$

9. $\dfrac{2}{x-2}-\dfrac{5}{x+2}=\dfrac{2}{x^2-4}.$

10. $\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}=\dfrac{4}{x^2-1}.$

11. $\dfrac{2x+3}{3x-4}=\dfrac{4x+5}{6x-1}.$

12. $\dfrac{6x+7}{9}+\dfrac{7x-13}{6x+3}=\dfrac{2x+4}{3}.$

13. $\dfrac{a(b^2+x^2)}{bx}=ac+\dfrac{ax}{b}.$

14. $\dfrac{3x}{2x+3}-\dfrac{2x}{2x-3}=\dfrac{2x^2-5}{4x^2-9}.$

15. $\dfrac{x-1}{x-2}-\dfrac{x-2}{x-3}=\dfrac{x-5}{x-6}-\dfrac{x-6}{x-7}.$

16. $\dfrac{2(x-7)}{x^2+3x-28}+\dfrac{x-2}{x-4}=\dfrac{x+3}{x+7}.$

17. $\frac{2x+1}{2x-16}-\frac{2x-1}{2x+12}=\frac{9x+17}{x^2-2x-48}.$

18. $\frac{a-b}{x-c}+\frac{b-c}{x-a}=\frac{a-c}{x}.$

19. $(a^2+x)(b^2+x)=(ab+x)^2.$

20. $(x+a)^4-(x-a)^4-8ax^3+8a^4=0.$

21. $(x-1)^3+x^3+(x+1)^3=3x(x^2-1).$

22. $(x-1)(x+2)(x-3)=x^2(x-2)+2(x+4).$

248. Theorem. *If $x+\sqrt{y}=a+\sqrt{b}$, in which x and a are rational and $\sqrt{y}$ and $\sqrt{b}$ surd, the rational and surd terms are separately equal.*

DEM. Transposing, we have

$$x-a=\sqrt{b}-\sqrt{y}.$$

Since a rational quantity cannot equal a surd, this equation can be true only when

$$x-a=0 \text{ and } \sqrt{b}-\sqrt{y}=0,$$

whence $$x=a \text{ and } \sqrt{b}=\sqrt{y}.$$

249. Theorem. *If $x+\sqrt{-y}=a+\sqrt{-b}$, in which x and a are real and $\sqrt{-y}$ and $\sqrt{-b}$ imaginary, the real and imaginary terms are separately equal.*

DEM. Transposing, we have

$$x-a=\sqrt{-b}-\sqrt{-y}.$$

Since a real quantity cannot equal an imaginary one, this equation can be true only when

$$x-a=0 \text{ and } \sqrt{-b}-\sqrt{-y}=0,$$

whence $$x=a \text{ and } \sqrt{-b}=\sqrt{-y}.$$

250. Many equations containing surds become simple equations after being freed from surds and reduced. No general rule can be given for solving such equations, as different cases must

have different treatment. Much depends on the student's insight. When inspection does not suggest what steps will free the equation from surds, the student must resort to trial, being careful to preserve the equality of the members. We give here a few suggestions. Some of the operations, as will be shown in Art. 351, cause roots to be lost or extraneous roots to be introduced.

1. By making a surd term constitute one member of an equation, its radical sign will disappear when both members are raised to a power of the same degree as the surd. A repetition of the process (having treated the most complicated surd first and reduced as much as possible) will cause a second radical sign to disappear, and so on.

2. When a surd denominator is similar to the surd numerator of another fraction or to another term, it is usually best to multiply both members by this denominator.

3. It is sometimes best to rationalize a surd denominator, especially when it differs only by a sign from the numerator of the same fraction.

4. Sometimes a surd factor, or a factor containing a surd term, can be removed either from both terms of a fraction or from both members of the equation.

EXAMPLES LXIX*

Solve the following:

1. $\sqrt{x-32}=16-\sqrt{x}.$ **2.** $\dfrac{x}{\sqrt{b^2+x}-b}=c.$

3. $\sqrt{x}+\sqrt{x-7}=\dfrac{21}{\sqrt{x-7}}.$ **4.** $\dfrac{5x-9}{\sqrt{5x}+3}-1=\dfrac{\sqrt{5x}-3}{2}.$

5. $\sqrt{a+\sqrt{x}}+\sqrt{a-\sqrt{x}}=\sqrt{x}.$

6. $\sqrt{(1+a)^2+(1-a)x}+\sqrt{(1-a)^2+(1+a)x}=2a.$

7. $\sqrt{\{13+\sqrt{[7+\sqrt{(3+\sqrt{x})}]}\}}=4.$

8. $\sqrt{1+\sqrt{3+\sqrt{6x}}}=2.$

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* Most of the examples of this set are from Olney's University Algebra.

9. $\dfrac{\sqrt{6x}-2}{\sqrt{6x}+2}=\dfrac{4\sqrt{6x}-9}{4\sqrt{6x}+6}.$

10. $\dfrac{243+324\sqrt{3x}}{16x-3}=16x-8\sqrt{3x}+3.$

11. $\dfrac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}}=b.$

12. $\dfrac{ax-1}{\sqrt{ax}+1}=4+\dfrac{\sqrt{ax}-1}{2}.$

13. $\dfrac{3\sqrt{x}-4}{2+\sqrt{x}}=\dfrac{15+3\sqrt{x}}{40+\sqrt{x}}.$

14. $\dfrac{\sqrt{ax}+\sqrt{b}}{\sqrt{ax}-\sqrt{b}}=\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{b}}.$

15. $\dfrac{\sqrt{a}-\sqrt{a-\sqrt{a^2-ax}}}{\sqrt{a}+\sqrt{a-\sqrt{a^2-ax}}}=b.$

16. $\dfrac{\sqrt{m}+\sqrt{m-y}}{\sqrt{m}-\sqrt{m-y}}=\dfrac{1}{m}.$

17. $\dfrac{a+x+\sqrt{2ax+x^2}}{a+x-\sqrt{2ax+x^2}}=b^2.$

18. $\dfrac{\sqrt[3]{x+1}-\sqrt[3]{x-1}}{\sqrt[3]{x+1}+\sqrt[3]{x-1}}=\dfrac{1}{3}.$

19. $\sqrt[3]{a+x}+\sqrt[3]{a-x}=b.$

20. $\dfrac{1+x+\sqrt{2x+x^2}}{1+x-\sqrt{2x+x^2}}=a\dfrac{\sqrt{2+x}+\sqrt{x}}{\sqrt{2-x}-\sqrt{x}}.$

21. $\dfrac{\sqrt{3x+1}+\sqrt{3x}}{\sqrt{3x+1}-\sqrt{3x}}=4.$

22. $\dfrac{1}{\sqrt{a-x}+\sqrt{a}}+\dfrac{1}{\sqrt{a-x}-\sqrt{a}}=\dfrac{\sqrt{a}}{x}.$

23. $\dfrac{a+2x+\sqrt{a^2-4x^2}}{a+2x-\sqrt{a^2-4x^2}}=\dfrac{5x}{a}.$

24. $\dfrac{18(7x-3)}{2x+1}=\dfrac{250\sqrt{2x+1}}{3\sqrt{7x-3}}.$

PROBLEMS LEADING TO SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY

251. The solution of a problem by algebraic methods consists of two distinct parts: 1st. Forming the equation or equations, which consists in expressing the conditions of the problem in algebraic language. 2d. Solving the equation or equations.

252. In solving an algebraic problem involving but one unknown quantity, we usually proceed as follows:

1. We represent the unknown quantity by one of the final letters of the alphabet.

2. We form an equation by indicating the operations that would be necessary to verify the result if it were known.

3. We solve this equation.

253. While we usually represent the number sought by x, it is sometimes advantageous to represent it by some multiple of x. For example, if three numbers whose sum is 36 are in the ratio of 2, 3, and 4, we may avoid fractions by letting $2x$, $3x$, and $4x$ be the numbers, giving the equation $2x + 3x + 4x = 36$. In other cases it may be advantageous to represent by x some number (time, for example) from which the number sought (distance, for example) is readily found.

EXAMPLES LXX

1. A bicyclist made a trip at the rate of 13 miles an hour. His return by a road 10 miles longer at the rate of 16 miles an hour required 15 minutes more time. What was the first distance?

SOLUTION. Let $x =$ the first distance;

then $x + 10 =$ the second distance,

$\frac{x}{13} =$ the first time,

and $\frac{x+10}{16} =$ the second time.

Now, since this second time is longer than the first by $\frac{1}{4}$ of an hour,

$$\frac{x}{13} = \frac{x+10}{16} - \frac{1}{4}.$$

Clearing of fractions, $16x = 13x + 130 - 52.$

Transposing and collecting, $3x = 78.$

Therefore $x = 26.$

2. A can do in 20 days a piece of work which B can do in 12 days. A began the work, but after a time B took his place, and the whole work was finished in 14 days from the beginning. How long did each work?

SOLUTION. Let $x =$ the no. of days A worked;

then $14 - x =$ the no. of days B worked;

$\frac{1}{20} =$ the part of the work A did in 1 day,

$\frac{1}{12} =$ the part of the work B did in 1 day,

$\frac{x}{20} =$ the part of the work A did in x days,

$\frac{14 - x}{12} =$ the part of the work B did in $14 - x$ days.

Now the sum of what A and B did in their respective times was the whole work, or 1. Hence

$$\frac{x}{20} + \frac{14 - x}{12} = 1.$$

Clearing of fractions, $3x + 70 - 5x = 60.$

Transposing and collecting, $-2x = -10.$

Therefore $x = 5,$

and $14 - x = 9.$

3. A post is $\frac{1}{5}$ in the earth, $\frac{3}{7}$ in the water, and 13 feet in the air. Find the length of the post.

4. A post is $\frac{m}{n}$ in the earth, $\frac{p}{q}$ in the water, and a feet in the air. Find the length of the post.

5. If an outward trip is made at the rate of 30 miles an hour, the return trip at 18 miles an hour, and the whole time is 10 hours, what is the distance?

6. When a is taken from the numerator of a fraction whose numerator is b less than its denominator, the value of the fraction becomes $\frac{m}{n}$. What is the original fraction?

7. When 5 is taken from the numerator of a fraction whose numerator is 3 less than its denominator, the value of the fraction becomes $\frac{3}{7}$. What is the original fraction?

SUG. Let $a = 5$, $b = 3$, $m = 3$, and $n = 7$, and substitute in the formula obtained from the solution of the preceding example. We thus obtain at once from the general case the result for the special case.

8. A man having completed $\frac{2}{5}$ of his journey, finds that after traveling 30 miles farther only $\frac{3}{7}$ of the journey remains. Find the length of the journey.

9. A man rows down a river for 2 hours at a rate that would take him 4 miles an hour in still water; then, resting, he floats with the current for half an hour. He then rows back to the starting place in 3 hours. Find the rate of the current.

10. A grocer drew 14 gallons of syrup from a cask which had lost $\frac{1}{4}$ part by leakage, and found that $\frac{2}{5}$ remained. Find the capacity of the cask.

11. A and B can together do in 12 days as much work as A can do alone in 20 days. In how many days could B alone do the same amount of work?

12. A cistern can be filled by the first of three pipes in $1\frac{1}{2}$ hours, by the second in $2\frac{1}{3}$ hours, and by the third in 5 hours. In what time can they together fill the cistern?

13. A can do in 15 hours a piece of work which B can do in 25 hours. After A has worked for a certain time, B completes the job, working 9 hours longer than A. How many hours did A work?

14. A cistern which contains 2400 gallons can be filled in 15 minutes by three pipes, the first of which lets in 10 gallons per minute, and the second 4 gallons less than the third. How many gallons does the third let in per minute?

15. A tank can be filled by one of two pipes in 24 minutes, and by the other in 30 minutes, and emptied by a third in 20 minutes. In what time will the tank be filled if all are left open?

16. A can row 4 and B 3 miles an hour in still water. A is 14 miles farther upstream than B, and they row toward each other till they meet, 4 miles above B's starting place. Find the rate of the current.

17. On a sum of money borrowed, annual interest is paid at 5%. After a time \$200 is paid on the principal, and the interest on the remainder is reduced to 4%. By these changes the annual interest is lessened one third. What is the sum borrowed?

18. A grocer has two kinds of coffee, one at a cents, and the other at b cents a pound. How much of each must he take to make a mixture of n pounds at c cents a pound?

19. A grocer has two kinds of tea, one at 60 cents and the other at 90 cents a pound. How much of each must he take to make a mixture of 120 pounds at 80 cents a pound?

SUG. Solve by substituting in the formula obtained from the solution of the preceding example.

20. If, for a given distance, the rate is 3 miles an hour over the first half, what must it be over the second half to make the average rate 4 miles an hour? 5 miles an hour? 6 miles an hour?

21. The hind wheels and fore wheels of a carriage have circumferences 16 and 14 feet respectively. How far has the carriage advanced when the fore wheels have made 51 revolutions more than the hind wheels?

22. A train leaves A at 11 A.M. for B, at the rate of 25 miles an hour. Another train leaves C at noon and runs through A to B at the rate of 35 miles an hour, arriving at B 24 minutes later than the first train. The distance from C to A being 21 miles, find the distance from A to B.

23. An artesian well supplies a manufactory. The water is drawn out each week day from 3 A.M. to 6 P.M. twice as rapidly as it runs into the well. If the well contained 2250 gallons of water on Monday morning and was just emptied at 6 P.M. a week from the following Thursday, how many gallons flowed into the well per hour?

24. Two men, A and B, 57 miles apart, travel toward each other, A at the rate of 6 miles an hour and B at the rate of 5 miles an hour, B starting 20 minutes later than A. How far will each have traveled when they meet?

25. Find the time between 3 and 4 o'clock when the hands of a watch are opposite each other.

SOLUTION. Let $x =$ the time past 3; then x is also the number of minute spaces passed over by the minute hand, and $\frac{x}{12}$ is the number passed over by the hour hand, since it moves $\frac{1}{12}$ as fast. Now, x in spaces is made up of

three parts, viz., 15 spaces from the XII mark to the III mark, $\frac{x}{12}$ from the III mark to where the hour hand is, and 30 from where the hour hand is to where the minute hand is; *i.e.*,

$$x = 15 + \frac{x}{12} + 30 = \frac{x}{12} + 45,$$

whence $$x = 49\tfrac{1}{11}.$$

Hence the time is 3 hours $49\frac{1}{11}$ minutes.

26. Find the time between 3 and 4 o'clock when the hands of a watch are together, and the time when they are at right angles to each other.

27. Find the time between 4 and 5 o'clock when the minute hand of a watch is 18 minute spaces ahead of the hour hand.

28. A bicyclist, at 12 miles an hour, went to meet another who started half an hour later, at 15 miles an hour, from a place 114 miles distant. In how many hours after the second started did they meet, and how far did each go?

29. The sum of the two digits of a number is 7. If 9 be subtracted from the number, the digits will be interchanged. What is the number?

30. The left-hand digit of a number composed of six digits is 1. If the 1 be removed to units' place, the other digits remaining in the same order as before, the new number will be 3 times the original number. Find the number.

Sug. Let x be the number exclusive of the left-hand digit.

31. Accommodation trains leave A for B at intervals of a hours, running m miles an hour. Express trains run from B to A at the rate of n miles an hour. At what intervals of time does an express train meet the accommodation trains?

32. The duty on a certain article is reduced from \$2.25 per hundredweight, and in consequence of this reduction the consumption is increased one half, but the revenue falls one third. Find the duty per hundredweight after the reduction.

33. A man walking from a town A to another B at the rate of 4 miles an hour, starts one hour before a coach which goes 12 miles an hour, and is picked up by the coach. On arriving at B, he observes that his coach journey lasted 2 hours. Find the distance from A to B.

34. A hare, 50 of her leaps ahead of a hound, takes 4 leaps to the hound's 3; but 2 of the hound's leaps equal 3 of the hare's. How many leaps must the hound take to catch the hare?

35. A bicyclist starts on the road with a message at a rate which will require 6 hours for its delivery. At the halfway point it is taken by another bicyclist who rides 3 miles an hour faster than the first, and thus the message is delivered half an hour earlier than it would have been had the first continued. Find the rate of the first bicyclist and the whole distance.

36. A can do in 30 days a piece of work which B can do in 20 days. A begins the work, but after a time B takes his place, and the whole work is finished 25 days from the beginning. How long did A work?

37. A can do in 20 days a piece of work which B can do in 30 days. A begins the work, but later B takes his place and finishes it, working 10 days longer than A. How long did A work?

38. A man starts on a bicycle ride at the rate of 10 miles an hour, intending to be back in 2 hours. Owing to a breakdown he walks back at the rate of 4 miles an hour, and finds himself $1\frac{1}{2}$ hours late. Find how far he went.

39. The capacity of the second of four casks is $\frac{3}{7}$ of the first, the third is $\frac{3}{4}$ of the second, the fourth is $\frac{16}{9}$ of the third, and the first holds 15 quarts more than the third and fourth. How many quarts does each hold?

40. A man invested part of \$ 2550 in 3% stocks and the rest in railroad shares of \$ 25 each, which pay annual dividends of \$ 1.00 per share. The stocks cost him \$ 81 on a hundred, and the railroad shares \$ 24 per share. His income from each source is the same. Find the number of railroad shares and the amount of each investment.

41. An express train runs 15 miles an hour faster than an accommodation train, and occupies $\frac{9}{14}$ as much time in running 120 miles. The express train loses at intermediate stations half as much time as does the accommodation train, and the latter loses as much time as it would require in running 20 miles. Find the rate of each train.

SECTION II — SIMULTANEOUS SIMPLE EQUATIONS WITH TWO UNKNOWN QUANTITIES

254. Simultaneous Equations are such as are to be satisfied for the same values of the unknown quantities. They express different relations of the same unknown quantities and arise from different conditions of the same problem.

If we have a single equation containing two unknown quantities, we may assume any value we please for one and find such value for the other as will satisfy the equation. Not so, however, if the values are to satisfy at the same time two different equations, such as come from different conditions of the same problem. If the equations are of the first degree, only one set of values will satisfy both. We are now concerned with the methods of finding these values.

255. Elimination is the process of deducing from a set of two or more simultaneous equations containing as many unknown quantities a new set in which the number of equations and the number of unknown quantities shall be diminished by at least one.

256. Three methods of elimination are in common use, viz., by **Addition or Subtraction**, by **Comparison**, and by **Substitution**.

257. The rules for elimination by these different methods are here given and illustrated. The student should be able to show, first, why these operations give true equations, and second, why they eliminate one of the unknown quantities.

After eliminating one of the two unknown quantities from two equations, the resulting equation is to be solved for the remaining unknown quantity. Usually the other is found by substituting this value in the simpler of the two given equations, and solving the resulting equation.

258. Prob. *To eliminate by Addition or Subtraction.*

RULE. *1st. If the coefficients of the unknown quantity to be eliminated are not numerically the same, make them so by multiplying each by that number which will give for the product their* **l. c. m.**

2d. If the signs of these coefficients are unlike, add the equations; if alike, subtract one equation from the other.

ILLUSTRATIVE EXAMPLE

Solve $\begin{cases} 6x + 7y = 29, \\ 4x - 3y = 4. \end{cases}$

Multiplying the first by 2 and the second by 3 and subtracting,

$$\begin{array}{r} 12x + 14y = 58 \\ \underline{12x - 9y = 12} \\ 23y = 46 \\ \therefore\ y = 2. \end{array}$$

Substituting this value of y in the second of the given equations and solving for x,

$$4x - 6 = 4,$$
$$4x = 10,$$
$$x = 2\tfrac{1}{2}.$$

Had we multiplied the first equation by 3 and the second by 7 and added, y would have been eliminated.

After finding the value of one of the unknown quantities, the other should be found by *substituting this in the equation containing the smallest numbers.*

259. Prob. *To eliminate by Comparison.*

RULE. *1st. Find from each equation the value of the same unknown quantity in terms of the other and known quantities.*

2d. Place these two values equal.

ILLUSTRATIVE EXAMPLE

Solve $\begin{cases} 7x + 11y = 68, \\ 9x - 4y = 33. \end{cases}$

From the first, $$x = \frac{68 - 11y}{7}.$$

From the second, $$x = \frac{33 + 4y}{9}.$$

Hence, $$\frac{68 - 11y}{7} = \frac{33 + 4y}{9},$$

$$612 - 99y = 231 + 28y,$$
$$-127y = -381,$$
$$y = 3.$$

Substituting this value of y in the second of the given equations and solving for x,

$$9x - 12 = 33,$$

$$9x = 45,$$

$$x = 5.$$

260. Prob. *To eliminate by Substitution.*

Rule. *1st. Find from one of the equations the value of one of the unknown quantities in terms of the other and known quantities.*

2d. Substitute this value for the same unknown quantity in the other equation.

ILLUSTRATIVE EXAMPLE

$$\text{Solve} \begin{cases} 7x - 2y = 1, & (1) \\ 3x + 5y = 59. & (2) \end{cases}$$

From (1), $$y = \frac{7x - 1}{2}. \quad (3)$$

Substituting this value of y in (2) and solving for x,

$$3x + 5\left(\frac{7x-1}{2}\right) = 59,$$

$$6x + 35x - 5 = 118,$$

$$41x = 123,$$

$$x = 3.$$

Substituting this value of x in (3),

$$y = \frac{21 - 1}{2} = 10.$$

SOME PRACTICAL SUGGESTIONS

261. 1. When using the method by Addition or Subtraction, if one of the coefficients of one of the unknown quantities is itself the **l. c. m.** of these coefficients, eliminate that unknown quantity, inasmuch as this requires that only one of the equations be multiplied. Otherwise, eliminate the unknown quantity whose coefficients require the smallest multipliers to make them the same.

2. After finding the value of one of the unknown quantities, the other should be found by substituting this in the equation containing the smallest numbers, or, in case of literal equations, the simplest coefficients.

3. When either of the equations, after reduction, contains in more than one term the unknown quantity to be eliminated, it is not expedient to use the method by Addition or Subtraction, inasmuch as no monomial multiplier will make the coefficients alike.

4. When the unknown quantity to be eliminated occurs in a monomial denominator in both equations, do not clear of fractions before eliminating. Such equations may thus be solved by the methods for simple equations, although they are of the second degree.

EXAMPLES LXXI

Solve the following, using all the different methods of elimination:

1. $\begin{cases} 3x - 4y = 2, \\ 7x - 9y = 7. \end{cases}$

2. $\begin{cases} 2x + 7y = 41, \\ 3x + 4y = 42. \end{cases}$

3. $\begin{cases} 11x - 7y = 37, \\ 8x + 9y = 41. \end{cases}$

4 $\begin{cases} 5x - 3y = 4, \\ 7x - 12y = -10. \end{cases}$

5. $\begin{cases} 3x - 4y = 18, \\ 3x + 2y = 0. \end{cases}$

6. $\begin{cases} 2x - 5y = -21, \\ 13x - 4y = 120. \end{cases}$

7. $\begin{cases} 15x + 7y = 29, \\ 9x + 15y = 39. \end{cases}$

8. $\begin{cases} 22x + 15y = 9, \\ 18x + 25y = 71. \end{cases}$

9. $\begin{cases} 18x - 20y = 44, \\ 17x - 15y = 26. \end{cases}$

10. $\begin{cases} 17x - 69y = -103, \\ 14x - 13y = -41. \end{cases}$

11. $\begin{cases} \dfrac{2x}{3} + \dfrac{3y}{4} = -\dfrac{7}{2}, \\ \dfrac{x}{4} - \dfrac{2y}{5} = \dfrac{11}{2}. \end{cases}$

12. $\begin{cases} \dfrac{3}{x-1} + \dfrac{4}{y-1} = 0, \\ \dfrac{5}{2x-3} - \dfrac{7}{2y+13} = 0. \end{cases}$

13. $\begin{cases} 10x - \dfrac{y-5}{7} = 11, \\ 8y - \dfrac{x+3}{4} = -17. \end{cases}$

14. $\begin{cases} \dfrac{6+x-y}{1-x-y} = -\dfrac{7}{4}, \\ 2x + 3y = -1. \end{cases}$

15. $\begin{cases} \dfrac{\dfrac{2x}{3}-\dfrac{5y}{12}}{\dfrac{7}{4}}-\dfrac{\dfrac{3x}{2}-\dfrac{y}{3}}{\dfrac{23}{2}}=2, \\ \dfrac{x-y}{x+y}=\dfrac{1}{5}. \end{cases}$

16. $\begin{cases} \dfrac{x-3y}{2}-\dfrac{y-3x}{2}=8, \\ \dfrac{\frac{1}{5}x+\frac{3}{4}y}{\frac{1}{2}x-\frac{11}{3}y}=-\dfrac{7}{30}. \end{cases}$

17. $\begin{cases} ax+by=c, \\ a'x+b'y=c'. \end{cases}$

18. $\begin{cases} ax-by=2\,ab, \\ 2\,bx+2\,ay=3\,b^2-a^2. \end{cases}$

19. $\begin{cases} (b+c)x+(b-c)y=2\,ab, \\ (a+c)x-(a-c)y=2\,ac. \end{cases}$

20. $\begin{cases} (a+b)x-(a-b)y=3\,ab, \\ (a-b)x-(a+b)y=ab. \end{cases}$

21. $\begin{cases} \dfrac{10}{x}-\dfrac{9}{y}=4, \\ \dfrac{8}{x}-\dfrac{15}{y}=\dfrac{9}{2}. \end{cases}$

22. $\begin{cases} \dfrac{5}{3x}-\dfrac{7}{y}=\dfrac{29}{9}, \\ \dfrac{3}{x}+\dfrac{5}{4y}=-\dfrac{9}{8}. \end{cases}$

23. $\begin{cases} \dfrac{2}{ax}+\dfrac{3}{by}=5, \\ \dfrac{5}{ax}-\dfrac{3}{by}=2. \end{cases}$

24. $\begin{cases} \dfrac{a}{bx}+\dfrac{b}{ay}=0, \\ \dfrac{b}{ax}+\dfrac{a}{by}=\dfrac{b^2}{a^2}-\dfrac{a^2}{b^2}. \end{cases}$

25. $\begin{cases} \dfrac{a}{bx}+\dfrac{b}{ay}=a+b, \\ \dfrac{b}{x}+\dfrac{a}{y}=a^2+b^2. \end{cases}$

26. $\begin{cases} \dfrac{a+b}{x}-\dfrac{1}{ay}=-\dfrac{b}{a}, \\ \dfrac{ab-b^2}{x}+\dfrac{1}{y}=\dfrac{a^2+3\,ab}{a+b}. \end{cases}$

PROBLEMS LEADING TO SIMPLE EQUATIONS WITH TWO UNKNOWN QUANTITIES

EXAMPLES LXXII

1. The daily pay of 5 men and 4 boys is \$19, and the daily pay of 2 men exceeds that of 3 boys by \$3. What is the daily pay of each man and boy?

SOLUTION. Let $x =$ the daily pay of each man,

and $y =$ the daily pay of each boy;

then $5\,x + 4\,y = 19$,

and $2\,x - 3\,y = 3$.

From these equations we find $x = 3$ and $y = 1$.

2. The sum of the two digits of a number is 13, and the number diminished by that formed by reversing the digits is 27. Find the number.

SOLUTION. Let $x =$ the digit in tens' place,

and $y =$ the digit in units' place.

Then $10x + y =$ the number,

and $10y + x =$ the number formed by reversing the digits.

Hence, $x + y = 13,$

and $10x + y - (10y + x) = 27.$

From these equations we find $x = 8$ and $y = 5$.

Hence the number is $10 \times 8 + 5 = 85$.

3. An income of \$120 a year is derived from a sum of money invested, partly in $3\frac{1}{2}$ per cent stock and partly in 4 per cent stock. If the stock be sold when the first is at 108 and the second at 120, the sum realized will be \$3672. Find each investment.

SOLUTION. Let $x =$ the $3\frac{1}{2}$ per cent stock,

and $y =$ the 4 per cent stock.

Then $\frac{3\frac{1}{2}x}{100} =$ income from the first,

$\frac{4y}{100} =$ income from the second,

$\frac{108x}{100} =$ selling price of the first,

and $\frac{120y}{100} =$ selling price of the second.

Hence, $$\frac{3\frac{1}{2}x}{100} + \frac{4y}{100} = 120,$$

and $$\frac{108x}{100} + \frac{120y}{100} = 3672.$$

From these equations we find $x = 2400$ and $y = 900$.

4. One kind of wine is worth 72 cents a quart, and another 40 cents. How much of each must be put into a mixture of 50 quarts that shall be worth 60 cents a quart?

5. If 6 pounds of sugar and 10 pounds of tea cost \$6.30, and, at the same price, 10 pounds of sugar and 6 pounds of tea cost \$4.10, what is the price of each per pound?

6. A farmer bought 120 acres of land for \$13,200, paying \$80 an acre for part of it, and \$120 an acre for the remainder. Find the number of acres in each part.

7. If 5 be added to the numerator of a certain fraction, its value becomes $\frac{5}{3}$; and if 5 be subtracted from its denominator, its value becomes $\frac{5}{2}$. Find the fraction.

8. A crew that can row 12 miles an hour downstream finds that it takes twice as long to row a given distance upstream. Find the rate of the current and the rate of the crew in still water.

9. A certain number is equal to 4 times the sum of its two digits, and if 18 be added to it, the digits will be reversed. What is the number?

10. A and B can do a piece of work in 9 hours. After working together 7 hours, B finishes the work in 5 hours more. In how many hours could each do the work?

11. In an informal ballot a resolution was adopted by a majority of 10 votes; but in the formal ballot one fourth of those who had before voted for it voted against it, and the resolution was lost by a majority of 6 votes. How many voted each way in the formal ballot?

12. When a is added to the greater of two numbers, the sum is m times the less; but when b is added to the less, the sum is n times the greater. Find the numbers.

13. When 4 is added to the greater of two numbers, the sum is $3\frac{1}{4}$ times the less; but when 8 is added to the less, the sum is $\frac{1}{2}$ the greater. Find the numbers by substituting in the results of the preceding example.

14. If two trains, 100 miles apart, approach each other, they will meet in 2 hours; but if they run in the same direction, the slower train leading, they will be together in 10 hours. What is the rate of each train?

15. A father has two sons, one 4 years older than the other. Six years ago the father's age was 6 times the joint ages of his sons, while 2 years hence his age will be twice the joint ages of his sons. Find the age of each.

16. A and B can together do a certain work in 30 days; at the end of 18 days, however, B is called off, and A finishes it alone in 20 days more. Find the time in which each could do the work.

17. A man rows 30 miles down a river and then back to the starting place, his whole time being 12 hours. He finds that he can row 5 miles with the stream in the same time as 3 against it. Find his time down and up respectively.

18. In an alloy of silver and copper $\frac{1}{m}$ of the whole $+p$ ounces was silver, and $\frac{1}{n}$ of the whole $-q$ ounces was copper. How many ounces were there of each?

19. A man invests $5100, partly in $3\frac{1}{2}$ per cent stock at 90 and partly in 4 per cent stock at 120, and receives from the two investments $185 a year. How many shares of each stock does he buy?

20. A man sculls in 1 hr. 20 min. a certain distance down a stream which runs at the rate of 4 miles an hour. In returning it takes him 4 hr. 15 min. to reach a point 3 miles below his starting place. How far did he scull down the stream, and at what rate could he scull in still water?

21. A tank is supplied by two pipes. If the first be opened 6 minutes and the second 7 minutes, the tank will be filled; or if the first be opened 3 minutes and the second 12 minutes, the tank will be filled. In what time will each pipe fill the tank?

22. A cistern is supplied by three pipes, two of which are of the same size. When they are all open, $\frac{5}{12}$ of the cistern is filled in 4 hours; but if one of the equal pipes be closed, $\frac{7}{9}$ of the cistern is filled in $10\frac{2}{3}$ hours. In how many hours would each pipe fill the cistern?

23. An alloy of tin and lead, weighing 40 pounds, loses 4 pounds when immersed in water. It is found that 10 pounds of tin lose 1.375 pounds when immersed in water, and 5 pounds of lead lose .375 pounds. How many pounds of each metal are in the alloy?

24. A man invests \$ 17,200 in 3 per cent bonds at 90 and 5 per cent bonds at 108, and his incomes from the two investments are the same. Find the amount of each investment.

25. A dairyman mixed with 100 quarts of morning's milk the night's milk, from which the cream had been skimmed, selling the mixture for 5 cents per quart and the cream for 20 cents per quart. By this fraudulent means he realized \$ 1.80 more than he would had he sold at 5 cents per quart both the morning's milk and the night's milk without skimming. How much skimmed milk and how much cream did he sell?

26. B has \$ 1000 more capital than A, invests it at one per cent more, and receives \$ 80 more income. C has \$ 500 more capital than B, invests it at one per cent more, and receives \$ 70 more income. Find A's capital and rate.

27. Two men, A and B, are employed on a piece of work. A works $1\frac{1}{2}$ days by himself, when B joins him, and they complete $\frac{8}{13}$ of the work $2\frac{1}{2}$ days later; they then find that 2 days more will be required for them to finish the work. In how many days could each do the work?

28. If a pieces of one kind of money make a dollar, and b pieces of another kind make a dollar, how many pieces of each kind must be taken to have c pieces in a dollar?

29. The distance from A to B along a railway is 70 miles, the first 10 miles being level, the next 35 sloping upward, and the rest level. A train starting from A runs half the distance in 62 minutes, and the whole distance in 1 hour 52 minutes. Find the rates of the train on the level ground and on the up grade.

30. Two cyclists, A and B, ride a race, the course being from P to Q, a distance of 12 miles, and back. A gives B a start of 15 minutes, and meets him on his return journey 1680 yards from Q, afterward winning the race by 1 minute. Find the rate of each cyclist, assuming it uniform.

31. Two men received \$ 96 for a piece of work which they could do together in 30 days. When half the work was done, one of them stopped 8 days and the other 4 days. They completed the work in $35\frac{1}{2}$ days from the beginning. How long would each require to do the work, and how many dollars should each receive?

32. A man in a rowboat is at a distance a from two barges at the instant when they are passing each other, one coming toward him and the other going away from him, the two having the same rate. Show that if b and b' are the distances he rows before meeting one and overtaking the other,

$$\frac{2}{a}=\frac{1}{b}+\frac{1}{b'}.$$

33. A and B formed a partnership. A invested $20,000 of his own money and $5000 which he borrowed; B invested $22,000 of his own money and $8000 which he borrowed at the same rate of interest as was paid by A. At the end of a year A's share of the profits was $1750 more than the interest on his $5000, and B's was $2000 more than the interest on his $8000. What rate of interest did they pay, and what rate per cent did they realize on their investment?

34. An ingot of metal which weighs n pounds loses p pounds when weighed in water. This ingot is itself composed of two other metals, which we may call A and B. Now n pounds of A lose q pounds when weighed in water, and n pounds of B lose r pounds when weighed in water. How much of each metal does the original ingot contain?

35. A body moves with uniform velocity from A to B, 323 feet, and, without stopping, returns. A second body leaves B 13 seconds after the first leaves A and moves toward A with uniform velocity. The first body meets the second 10 seconds after the latter starts, and, in returning to A, overtakes the second body 45 seconds after the latter starts. Find the velocity of each body.

SECTION III — SIMULTANEOUS SIMPLE EQUATIONS WITH SEVERAL UNKNOWN QUANTITIES

262. Prob. *To solve several simultaneous simple equations with as many unknown quantities.*

RULE. *1st. Eliminate the same unknown quantity from different pairs of the given equations, thus forming a set of equations independent of this unknown quantity, and one less in number than the given equations.*

2d. From these, in like manner, eliminate another unknown quantity, and so continue till an equation with but one unknown quantity is found.

3d. Find the value of this unknown quantity and substitute it in one of the two equations of the next preceding set. Solve this equation and substitute the two values now found in one of the three equations of the next preceding set. Continue this process till all the unknown quantities are determined.

263. Sch. 1. It is usually best to combine the equation having the smallest coefficients with each of the other equations of the same set.

264. Sch. 2. If any equation of any set does not contain the unknown quantity we are eliminating, this equation is written unchanged in the next set.

EXAMPLES LXXIII

Solve the following:

1. $\begin{cases} 3x + 4y - z = 8, \\ x + 2y - 4z = -7, \\ 2x - 5y - 7z = -29. \end{cases}$

2. $\begin{cases} 2x + 3y + z = 17, \\ 2x + 2y + z = 14, \\ x + 3y + 2z = 19. \end{cases}$

3. $\begin{cases} 2x + 3y - 4z = 8, \\ 3x - 4y + 2z = 3, \\ 4x - 2y - 3z = 5. \end{cases}$

4. $\begin{cases} 12x - 4y + z = 3, \\ x - y - 2z = -1, \\ 5x - 2y = 0. \end{cases}$

5. $\begin{cases} 2x - 3y = 4, \\ 4x - 3z = 2, \\ 4y + 2z = -3. \end{cases}$

6. $\begin{cases} x + \dfrac{y+z}{2} = 85, \\ y + \dfrac{x+z}{3} = 85, \\ z + \dfrac{x+y}{4} = 85. \end{cases}$

7. $\begin{cases} w + 3x - y - z = 7, \\ 2w - 2x + y + 3z = 8, \\ 3w - x + y - 4z = 8, \\ 4w + x - y - 2z = 7. \end{cases}$

8. $\begin{cases} 2x + 3y + z = 23, \\ 2u + 3x + y = 25, \\ u + x + 3z = 24, \\ 3u + 2y + 2z = 36. \end{cases}$

9. $\begin{cases} u - 2x = -13, \\ x - 3y = 13, \\ y - 4z = 5, \\ z - 5u = 23. \end{cases}$

10. $\begin{cases} u + 5x - 7y + z = 2, \\ 2u + 7x - 3y - 3z = 2, \\ 4x - 2y = 2, \\ x + 5y - 2z = 2. \end{cases}$

Although the following are not of the first degree with reference to x, y, and z, they are of the first degree with reference to the reciprocals of these quantities. We may, therefore, regard $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$ as the unknown quantities and solve accordingly, *i.e.*, eliminate without first clearing of fractions.

11. $\begin{cases} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3, \\ \frac{1}{x} - \frac{1}{y} + \frac{1}{z} = 2, \\ \frac{1}{x} + \frac{1}{y} - \frac{1}{z} = 1. \end{cases}$

12. $\begin{cases} \frac{2}{x} + \frac{1}{y} - \frac{3}{z} = 0, \\ \frac{2}{y} - \frac{3}{z} = -2, \\ \frac{1}{x} + \frac{1}{z} = \frac{4}{3}. \end{cases}$

13. $\begin{cases} \frac{1}{x} + \frac{1}{y} = \frac{2}{3}, \\ \frac{1}{x} + \frac{1}{z} = \frac{3}{4}, \\ \frac{1}{y} + \frac{1}{z} = \frac{5}{6}. \end{cases}$

14. $\begin{cases} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 9, \\ \frac{2}{x} - \frac{3}{y} + \frac{4}{z} = 11. \\ \frac{5}{x} + \frac{2}{y} - \frac{3}{z} = 4. \end{cases}$

15. $\begin{cases} \frac{a}{x} + \frac{b}{y} = 1, \\ \frac{b}{y} + \frac{c}{z} = 1, \\ \frac{c}{z} + \frac{a}{x} = 1. \end{cases}$

16. $\begin{cases} \frac{1}{x} + \frac{3}{2y} = \frac{9}{5}, \\ \frac{1}{y} + \frac{4}{3z} = \frac{5}{3}, \\ \frac{1}{z} + \frac{5}{4x} = \frac{7}{4}. \end{cases}$

In solving the next two examples, divide each equation by the unknown quantities in the second member, and then eliminate fractional terms.

17. $\begin{cases} y + z = 2yz, \\ x + z = 3xz, \\ x + y = 4xy. \end{cases}$

18. $\begin{cases} yz + 4xz - 3xy = 2xyz, \\ 3yz - 2xz + 6xy = 4xyz, \\ 5yz - 6xz - 3xy = xyz. \end{cases}$

In solving the next two examples, let xyz constitute one member of each equation, divide each equation by xyz, and then eliminate fractional terms.

19. $xyz = 2(xz + xy - yz) = 3(xz + yz - xy) = 4(xy + yz - xz).$

20. $xyz = a(yz - xz - xy) = b(xz - xy - yz) = c(xy - yz - xz).$

When one of the unknown quantities is wanting in each of the given equations, and all the coefficients (after simplifying the equations) are unity, the work may be made very short by adding all the equations, dividing by the common coefficient, and then subtracting from the resulting equation each of the given equations in turn. Likewise, when all of the unknown quantities are present in each equation, and all of the coefficients (after simplifying the equations) except one are unity, and this one coefficient is repeated with successive unknown quantities in successive equations, the work may be made very short by adding all the equations, dividing by the common coefficient, and then subtracting the resulting equation from each of the given equations in turn. Solve in this manner the following:

21. $\begin{cases} x + y = 9, \\ x + z = 10, \\ y + z = 11. \end{cases}$

22. $\begin{cases} w + x + y = 7, \\ w + x + z = 6, \\ w + y + z = 9, \\ x + y + z = 8. \end{cases}$

23. $\begin{cases} w + x + y = 3, \\ w + x + z = -4, \\ w + y + z = 2, \\ x + y + z = -1. \end{cases}$

24. $\begin{cases} w + x + y = 0, \\ w + x + z = 6, \\ w + y + z = -1, \\ x + y + z = -2. \end{cases}$

25. $\begin{cases} v + w + x + y = 10, \\ v + w + x + z = 11, \\ v + w + y + z = 12, \\ v + x + y + z = 13, \\ w + x + y + z = 14. \end{cases}$

26. $\begin{cases} u + v + w + x + y = 3, \\ u + v + w + x + z = 8, \\ u + v + w + y + z = 4, \\ u + v + x + y + z = 9, \\ u + w + x + y + z = 6, \\ v + w + x + y + z = 5. \end{cases}$

27. $\begin{cases} 3w + x + y + z = 14, \\ w + 3x + y + z = 12, \\ w + x + 3y + z = 16, \\ w + x + y + 3z = 18. \end{cases}$

28. $\begin{cases} w + x + y + 4z = -9, \\ w + x + 4y + z = 3, \\ w + 4x + y + z = -6, \\ 4w + x + y + z = 12. \end{cases}$

29. $\begin{cases} 2w+2x+2y+3z=39, \\ 2w+2x+3y+2z=37, \\ 2w+3x+2y+2z=35, \\ 3w+2x+2y+2z=33. \end{cases}$

30. $\begin{cases} w+5x+y+z=22, \\ w+x+y+5z=26, \\ 5w+x+y+z=18, \\ w+x+5y+z=-18. \end{cases}$

31. $\begin{cases} v+w+x+y+2z=52, \\ v+w+x+2y+z=50, \\ v+w+2x+y+z=48, \\ v+2w+x+y+z=46, \\ 2v+w+x+y+z=44. \end{cases}$

32. $\begin{cases} v+w+6x+y+z=46, \\ 6v+w+x+y+z=36, \\ v+w+x+6y+z=1, \\ v+6w+x+y+z=-4, \\ v+w+x+y+6z=31. \end{cases}$

PROBLEMS LEADING TO SIMPLE EQUATIONS WITH SEVERAL UNKNOWN QUANTITIES

EXAMPLES LXXIV

1. The sum of the three digits of a number is 12; the digit in the tens' place is $\frac{1}{5}$ the sum of the other two, and the number expressed by the two left-hand digits is 7 times the digit in units' place. Find the number.

2. For $8 I can buy 2 pounds of tea, 10 pounds of coffee, and 20 pounds of sugar; or 2 pounds of tea, 5 pounds of coffee, and 30 pounds of sugar; or 3 pounds of tea, 5 pounds of coffee, and 10 pounds of sugar. What are the prices?

3. Three cities, *A*, *B*, and *C*, not in the same straight line, are connected by straight roads. The distance from *A* to *C* by way of *B* is 82 miles, from *B* to *A* by way of *C* is 97 miles, and from *C* to *B* by way of *A* is 89 miles. Find the distances between the cities.

Sug. In solving the equations, proceed as directed for examples 21 to 26 of the preceding set.

4. The total capacity of 3 casks is 1440 quarts. Two of them are full, and one is empty. To fill the empty one requires the contents of the first and $\frac{1}{5}$ the contents of the second, or the contents of the second and $\frac{1}{3}$ the contents of the first. Find the capacity of each cask.

5. A farm was rented for 2 years for a fixed money payment and 400 bushels of wheat and barley. The first year wheat was 70 cents per bushel and barley 50 cents, and the entire rent was $ 1000. The second year wheat was 50 cents per bushel and barley 45 cents, and the entire rent was $ 950. Find the amounts of money, wheat, and barley paid as rent each year.

6. When 3 partners began business, A had $ 2000 more than twice as much capital as B, and C had $ 500 less than A and B together. The first year A gained as much as B's capital, B gained as much as A's capital, and C gained as much as A's and B's capital together, whereupon each had the same sum. Find how much each had at first, and interpret the results.

7. The capacity of 3 casks is 344 gallons, and all are full. Fifty gallons are used from the first; then $\frac{1}{3}$ of what is in the second is poured into the first, and $\frac{1}{5}$ of what is in the third is poured into the second. After these changes, the first contains 10 gallons more than the second, and the second 10 gallons more than the third. Find the capacity of each cask.

8. In walking along a street on which electric cars are running at equal intervals from both ends, I observe that I am overtaken by a car every 12 minutes, and that I meet one every 4 minutes. What are the relative rates of myself and the cars, and at what intervals of time do the cars start?

9. Four towns, A, B, C, and D, connected by rail, are at the vertices of a quadrilateral. A commercial traveler, in making the rounds of these towns, observes that in going from A by way of B and C to D, the fare, at 2 cents per mile, is $ 1.22, and in going from A by way of D and C to B, it is $ 1.10; while from A by way of B to C, it is the same as from A by way of D to C, and from B by way of A to D, it is 40 cents less than from B by way of C to D. Find the distances.

10. A, B, C, and D engage to do a certain work. A and B can do it in 12 days, A and D in 15 days, and D and C in 18 days. B and C begin the work together, after 3 days are joined by A, and after 4 days more by D. Then all working together they finish it in 2 days more. How long would each have required to do the entire work?

CHAPTER XII

INEQUALITIES

265. An Inequality is an assertion by means of a mathematical symbol that two expressions have different values. This symbol is the *Sign of Inequality* $>$ or $<$, read "greater than" or "less than," according as the opening is to the left or right.

266. The Members of an Inequality are the expressions connected by the sign of inequality, the one on the left being called the **First Member**, and the one on the right the **Second Member.**

267. Of two positive quantities, the greater is the one which is numerically the greater; while of two negative quantities the greater is the one which is numerically the less. A negative quantity is less than a positive quantity, regardless of their numerical values. In general, when $a - b$ is positive, $a > b$, and when $a - b$ is negative, $a < b$.

268. Two inequalities *exist in the same sense* or *exist in an opposite sense,* according as their signs of inequality are alike or opposite.

269. Theorem. *An inequality will continue to exist in the same sense,*

1st. By adding equals to both members, or subtracting equals from both members.

2d. By multiplying or dividing both members by equal positive quantities.

3d. By raising both members to the same odd power.

4th. By raising both members to the same even power, if both members are positive.

5th. By extracting the same root of both members, if when the index is even, only the positive roots be compared.

Dem. Each of the operations of the 1st and 2d increases or decreases both members alike, either by the same amount or in the same ratio, and does not change the signs of the members. Therefore, that member which was greater before any of the operations named is greater after.

Since the same power or root of the numerically greater of two quantities is numerically the greater, and the operations of the 3d, 4th, and 5th do not change the signs of the members, that member which was greater before any of the operations named is greater after.

270. Cor. 1. *An inequality may be cleared of fractions without changing the sense in which it exists.*

This is simply multiplying both members by the same positive quantity.

271. Cor. 2. *A term, by changing its sign, may be transposed from one member of an inequality to the other without changing the sense in which it exists.*

This is simply subtracting the same quantity from, or adding the same quantity to, both members.

272. Theorem. *An inequality will be made to exist in an opposite sense,*

1st. By changing the signs of both members.

2d. By multiplying or dividing both members by the same negative quantity.

3d. By raising both members to the same even power, if both members are negative.

4th. By comparing the negative even roots.

Dem. The members are affected numerically as in the preceding demonstration, but the signs are changed. Therefore, that member which was greater before any of the operations named is less after.

273. Theorem. *If two or more inequalities exist in the same sense,*

1st. The sums will exist in the same sense as the given inequalities.

2d. The products will exist in the same sense as the given inequalities, if their members are all positive.

Dem. In the 1st case the greater member of any one of the inequalities is increased by a greater amount or decreased by a less amount than the less member; and in the 2d case the greater member is increased in a greater ratio or diminished in a less ratio than the less member.

274. Caution. The student is cautioned against taking the differences or the quotients of two inequalities that exist in the same sense, inasmuch as the differences and quotients will exist in the same sense as, or the opposite sense from, the given inequalities, according to the order of subtracting or dividing, or will give, with certain relations, an equation.

275. Note. In determining which of two expressions is the greater it is sometimes best to place both signs, $\gtrless$, read "greater or less," between them and then operate on the inequality in such ways as not to change the sense, until it becomes apparent which is the greater member. This will show which of the two signs to use in the original inequality.

EXAMPLES LXXV

It is understood that the letters of each example are positive and unequal.

1. Which is greater, the sum of the squares of any two quantities or twice their product?

Solution. We have

$$a^2 + b^2 \gtrless 2\,ab.$$

Transposing, $$a^2 - 2\,ab + b^2 \gtrless 0,$$

or $$(a - b)^2 \gtrless 0.$$

But $(a - b)^2$ is necessarily plus and, therefore, greater than 0. Hence, since the operations leave the inequality in the same sense as the first,

$$a^2 + b^2 > 2\,ab.$$

2. Which is greater, half the sum of two quantities (their arithmetical mean) or the square root of their product (their geometrical mean)?

SOLUTION. We have $$\frac{a+b}{2} \gtrless \sqrt{ab}.$$

Squaring, etc., $$a^2 + 2\,ab + b^2 \gtrless 4\,ab,$$

$$a^2 - 2\,ab + b^2 \gtrless 0,$$

$$(a-b)^2 \lessgtr 0.$$

But $(a-b)^2 > 0$, because the square of any quantity is plus. Hence, since the operations leave the inequality in the same sense as the first,

$$\frac{a+b}{2} > \sqrt{ab}.$$

3. Show that $a^2 + b^2 + c^2 > ab + ac + bc$.

SOLUTION. By Ex. 1, $$a^2 + b^2 > 2\,ab,$$

$$a^2 + c^2 > 2\,ac,$$

$$b^2 + c^2 > 2\,bc.$$

Adding and dividing by 2, $a^2 + b^2 + c^2 > ab + ac + bc$.

4. Which is greater, $\dfrac{a^2 - x^2}{a^2 + x^2}$ or $\dfrac{a-x}{a+x}$, if $x < a$?

5. Show that any fraction plus its reciprocal is greater than 2.

6. If a, b, c are such that the sum of any two is greater than the third, show that $a^2 + b^2 + c^2 < 2(ab + ac + bc)$.

7. If $a^2 + b^2 + c^2 = 1$, and $m^2 + n^2 + r^2 = 1$, show whether $am + bn + cr$ is greater or less than 1.

8. Show that

$$(a + b - c)^2 + (a + c - b)^2 + (b + c - a)^2 > ab + bc + ac.$$

9. Which is greater, $a^3 + b^3$ or $a^2b + ab^2$?

10. Prove that $(ab + xy)(ax + by) > 4\,abxy$.

11. Which is greater, $2\,x^3$ or $x + 1$, if $x > 1$?

12. Find the limits of x determined by the conditions

$$\frac{x-7}{9} + \frac{x-4}{5} > 9, \text{ and } \frac{x-10}{6} < \frac{x+5}{9} - \frac{1}{6}.$$

13. The double of a number diminished by 5 is greater than 25, and the triple of the number diminished by 7 is less than the double increased by 13. Find the limits of the number.

14. A man wishes to make a purchase for \$ 14, but has not enough money. If he borrows one third as much money as he now has, he will be able to make the purchase and have more money left than he now lacks. How much money has he?

15. The daily pay roll of a contractor, who pays masons \$ 4.40 a day and carpenters \$ 3.60 a day, is between \$ 104 and \$ 112, and there are 3 more masons than carpenters. Find the number of each.

16. A is 24 years old and B is 15. What is the shortest time after which A's age will be less than $1\frac{2}{5}$ times B's age?

17. If only 5 pupils be seated on each bench in a recitation room containing fewer than 6 benches, 4 pupils will be without seats; but if 6 pupils be seated on each bench, some seats will be unoccupied. Find the number of benches.

18. The sum of two whole numbers is 25. If the greater be divided by the less, the quotient will be less than $3\frac{1}{2}$; and if the less be divided by the greater, the quotient will be greater than $\frac{1}{5}$. What are the numbers?

CHAPTER XIII

RATIO, PROPORTION, AND VARIATION

SECTION I — RATIO

276. **Ratio** is the relative magnitude of two quantities of the same kind, and is measured by the quotient of the first by the second.

The ratio of two quantities is expressed either by writing a colon between them or by writing them in the fractional form.

Thus the ratio of a to b is written either $a : b$ or $\frac{a}{b}$.

277. **The Antecedent** or **First Term** of a ratio is the first of the two quantities compared, and **the Consequent** or **Second Term** is the second.

278. A ratio is a **Ratio of Greater Inequality, Less Inequality,** or **Equality,** according as it is greater than, less than, or equal to, unity.

279. **The Duplicate, Sub-duplicate, Triplicate,** and **Sub-triplicate Ratios** of two quantities are the ratios of the squares, square roots, cubes, and cube roots of those quantities respectively.

280. **A Compound Ratio** is the ratio of the products of the corresponding terms of two or more simple ratios.

281. **Theorem.** *1st. A ratio is not changed by multiplying or dividing both its terms by the same quantity.*

2d. A ratio is multiplied by multiplying its antecedent or dividing its consequent.

3d. A ratio is divided by dividing its antecedent or multiplying its consequent.

Since a ratio is simply a fraction, or an indicated division, these follow from Arts. 127 and 137.

282. Theorem. *A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by adding the same positive quantity to both its terms;* i.e., *any ratio is made more nearly equal to unity by adding the same positive quantity to both its terms.*

Dem. Let $m : n$, or $\frac{m}{n}$, be the given ratio, and a the quantity to be added to both its terms. Then we have

$$\frac{m}{n} \gtrless \frac{m+a}{n+a}. \tag{1}$$

Whichever member is greater will still be greater after both are multiplied by $n(n+a)$, giving

$$mn + am \gtrless mn + an. \tag{2}$$

Since mn is common to both members, the first member of (2) will be greater or less than the second, according as am is greater or less than an, or m greater or less than n. Hence the first member of (1) is greater or less than the second, according as m is greater or less than n.

EXAMPLES LXXVI

Find the values of the following ratios:

1. $x^2 - 7x + 10 : x - 2$. **2.** $x + 5 : x^2 + 3x - 10$.

3. $x^4 - y^4 : x^3 - x^2y + xy^2 - y^3$. **4.** $12(a-b)^2 : 8(a^2 - b^2)$.

5. $3x^3 - 2x^2 - 19x - 6 : 3x^3 + 4x^2 - 5x - 2$.

6. The duplicate ratio of $13 : 39$.

7. The sub-duplicate ratio of $121x^2 - 726x + 1089 : x^2 - 6x + 9$.

8. The triplicate ratio of $\frac{2}{3} : \frac{5}{6}$.

9. The sub-triplicate ratio of $729 : 1728$.

10. Is $a^2 - x^2 : a^2 + x^2$ greater or less than $a - x : a + x$?

11. Is $x^3 + y^3 : x^2 + y^2$ greater or less than $x^2 + y^2 : x + y$?

12. Is $x^3 - y^3 : (x+y)^2$ greater or less than $x^2 + xy + y^2 : x - y$, x being greater than y?

13. If 6 be added to each of two numbers that are in the ratio of $5:7$, the sums will be in the ratio of $7:9$. Find the numbers.

14. If 8 be subtracted from each of two numbers that are in the ratio of $2:5$, the remainders will be in the ratio of $2:9$. Find the numbers.

15. What must be subtracted from each term of $13:25$ to make it $1:3$?

16. A certain ratio becomes $\frac{3}{4}$ when 4 is added to both its terms, and $\frac{1}{2}$ when 2 is subtracted from both its terms. Find the ratio.

17. What quantity subtracted from each term of the duplicate ratio of $m:n$ will give the triplicate ratio of $m:n$?

18. If 5 gold coins and 30 silver ones are worth as much as 10 gold coins and 10 silver ones, what is the ratio of their values?

19. If the wages of 5 boys and 6 girls is $\frac{3}{4}$ of the wages of 6 boys and 9 girls for the same time, what is the ratio of their wages?

20. The sides of a triangle are in the ratio of 3, 4, and 5, and the perimeter is 480 yards. Find the sides.

21. The ratio of a father's age to his son's is $7:2$, and the father is 30 years older than the son. Find the age of each.

22. A fox makes 4 leaps while a hound makes 3; but 2 of the hound's leaps equal 3 of the fox's. Find relative rates of running.

23. If in working out road tax 6 men and 4 teams are counted as much as 10 men and 1 team, what is the ratio of wages for men and teams?

24. A bequest of \$900 was divided among three sons, and after their shares had increased by \$10, \$15, and \$20 respectively, the sums were in the ratio $4:5:6$. Find the shares.

25. The weights of two loads are in the ratio of 4 to 5. Parts of the loads in the ratio of 6 to 7 being removed, the remaining weights are in the ratio of 2 to 3, and the sum of the weights is then 10 tons. What were the weights at first?

26. In a college boat race, crew A pull 15 strokes to 14 strokes of crew B; but 28 strokes of crew B are as effective as 33 of crew A. Which is the faster crew, and in what ratio?

27. Find the gear of a bicycle whose wheels are d inches in diameter, whose front sprocket has m teeth and whose rear sprocket has n teeth, it being understood that the "gear" of a bicycle is the diameter of a wheel one revolution of which would advance it as far as one revolution of the pedal advances the bicycle.

SOLUTION. Let $x =$ the gear.

One revolution of a wheel whose diameter is x would advance the wheel by the amount of its circumference, πx.

The circumferences of the sprocket wheels are in the ratio of their number of teeth, $m : n$. Hence $\frac{m}{n} =$ the number of revolutions of the bicycle wheels to one revolution of the pedal, and $\frac{m}{n} \times \pi d =$ the distance the bicycle advances for one revolution of the pedal. Therefore,

$$\pi x = \frac{m}{n}\pi d,$$

whence

$$x = \frac{md}{n}.$$

28. Find, by substituting in the formula of the last example, the gear in each of the following cases:

(a) $d = 28$ in., $m = 18$, $n = 7$.

(b) $d = 28$ in., $m = 21$, $n = 7$.

(c) $d = 30$ in., $m = 20$, $n = 8$.

(d) $d = 28$ in., $m = 22$, $n = 8$.

SECTION II — PROPORTION

283. **A Proportion** is an equality of ratios.

The equality is indicated by the sign of equality or by the double colon. If the ratio $a : b$ equals the ratio $c : d$, the proportion may be written in any one of the three ways,

$$a : b :: c : d,\ a : b = c : d, \text{ or } \frac{a}{b} = \frac{c}{d}.$$

In any form the proportion is read, "a is to b as c is to d."

284. Four quantities are **Directly Proportional** when the ratio of two of them is equal to the ratio of the other two taken in the same order.

Thus, the times being the same, any two distances are directly proportional to the corresponding rates.

285. Four quantities are **Inversely** or **Reciprocally Proportional** when the ratio of two of them is equal to the ratio of the other two taken in the inverse order.

Thus, the distances being the same, any two times are inversely or reciprocally proportional to the rates. If T is the time at the rate R, and t the time at the rate r, then

$$T : t :: r : R, \text{ or } T : t :: \frac{1}{R} : \frac{1}{r}.$$

The same relation is expressed by saying, "The times are in the inverse, or reciprocal, ratio of the rates," or "The times vary inversely, or reciprocally, as the rates."

286. **The Extremes** of a proportion are its first and last terms.

The Means of a proportion are its second and third terms.

287. When the means of a proportion are the same quantity, this quantity is called a **Mean Proportional** between the other two quantities, and the last term is called a **Third Proportional** to the other two quantities.

Thus, in $a : b :: b : c$, b is a mean proportional between a and c, and c is a third proportional to a and b.

288. **A Fourth Proportional** to three quantities is the fourth term of a proportion whose other three terms are the three quantities taken in their order.

289. A proportion is taken by **Inversion** when the terms of each ratio are written in inverse order.

290. A proportion is taken by **Alternation** when the means are interchanged, or when the extremes are interchanged.

291. A proportion is taken by **Composition** when the sum of the terms of each ratio is compared with either term of that ratio, the

same order being observed; or when the sum of the antecedents and the sum of the consequents are compared with either antecedent and its consequent.

292. A proportion is taken by **Division** if *difference* be substituted for *sum* in the last definition.

293. **A Continued Proportion** is a succession of equal ratios in which each consequent is the antecedent of the next ratio.

Thus, $a : b :: b : c :: c : d :: d : e$ is a continued proportion.

294. Theorem. *In any proportion the product of the extremes equals the product of the means.*

DEM. Let the proportion be

$$a : b :: c : d.$$

This is the same as $\frac{a}{b} = \frac{c}{d}$ (Art. 283).

Clearing of fractions, $ad = bc.$

295. Cor. 1. *A mean proportional between two quantities is equal to the square root of their product.*

For if $a : b :: b : c,$

$$b^2 = ac,$$

whence $b = \sqrt{ac}.$

296. Cor. 2. *Either extreme of a proportion equals the product of the means divided by the other extreme; and either mean equals the product of the extremes divided by the other mean.*

297. Theorem. *If the product of two quantities equals the product of two others, the quantities of one product may be made the extremes and the quantities of the other product the means of a proportion.*

DEM. Let $ad = bc.$

Dividing by bd, $\frac{a}{b} = \frac{c}{d};$

that is, $a : b :: c : d.$

Writing the first equation in the form

$$bc = ad,$$

and dividing by ac, $$\frac{b}{a} = \frac{d}{c};$$

that is, $$b : a :: d : c.$$

By dividing by other combinations of the letters other forms can be obtained; but in each the quantities of one product will be the extremes, and the quantities of the other product the means.

298. Theorem. *Proportionals result from taking a proportion* (*a*) *by inversion,* (*b*) *by alternation,* (*c*) *by composition,* (*d*) *by division,* (*e*) *by composition and division.*

Dem. Let the given proportion be

$$a : b :: c : d. \qquad (1)$$

Then proportionals result from taking this

(*a*) *By inversion,* $b : a :: d : c.$

From (1), $$\frac{a}{b} = \frac{c}{d}. \qquad (2)$$

Dividing unity by each member,

$$\frac{b}{a} = \frac{d}{c}; \qquad (3)$$

that is, $$b : a :: d : c.$$

(*b*) *By alternation,* $\begin{cases} a : c :: b : d, \\ d : b :: c : a. \end{cases}$

From (1), $$ad = bc. \qquad (4)$$

Dividing by cd, $$\frac{a}{c} = \frac{b}{d};$$

that is, $$a : c :: b : d.$$

Dividing (4) by ab, $$\frac{d}{b} = \frac{c}{a};$$

that is, $$d : b :: c : a.$$

(*c*) *By composition*, $\begin{cases} a+b:b::c+d:d, \\ a+b:a::c+d:c, \\ a+c:a::b+d:b, \\ a+c:c::b+d:d, \\ \text{and other forms.} \end{cases}$

Adding unity to both members of (2),

$$\frac{a}{b}+1=\frac{c}{d}+1,$$

or $$\frac{a+b}{b}=\frac{c+d}{d}; \tag{5}$$

that is, $$a+b:b::c+d:d.$$

Let the student demonstrate the other three forms given. Other forms are obtained by taking these four forms by inversion and by alternation.

(*d*) *By division*, $\begin{cases} a-b:b::c-d:d, \\ a-b:a::c-d:c, \\ a-c:a::b-d:b, \\ a-c:c::b-d:d, \\ \text{and other forms.} \end{cases}$

Subtracting unity from both members of (2),

$$\frac{a}{b}-1=\frac{c}{d}-1,$$

or $$\frac{a-b}{b}=\frac{c-d}{d}; \tag{6}$$

that is, $$a-b:b::c-d:d.$$

Let the student demonstrate the other three forms given. Other forms are obtained by taking these four forms by inversion and by alternation.

(*e*) *By composition and division*, $\begin{cases} a+b:a-b::c+d:c-d, \\ a+c:a-c::b+d:b-d, \\ \text{and other forms.} \end{cases}$

Dividing (5) by (6), $$\frac{a+b}{a-b}=\frac{c+d}{c-d};$$

that is, $$a+b:a-b::c+d:c-d.$$

Let the student demonstrate the other form given. Other forms are obtained by taking these two forms by inversion and by alternation.

299. Theorem. *If four quantities are in proportion, proportionals result from taking equimultiples, (a) of the terms of a couplet, (b) of the antecedents, (c) of the consequents, (d) of all the terms.*

DEM. Let the given proportion be

$$a : b :: c : d. \qquad (1)$$

Then proportionals result from taking equimultiples,

(*a*) *Of the terms of a couplet.*

From (1), $\frac{a}{b} = \frac{c}{d}.$

Multiplying numerator and denominator of the first member by m,

$$\frac{am}{bm} = \frac{c}{d};$$

that is, $am : bm :: c : d.$

Let the student demonstrate the other cases.

300. Theorem. *The same powers or roots of proportionals are proportional.*

Let the student demonstrate.

301. Theorem. *The products or the quotients of the corresponding terms of two (or more) proportions are proportional.*

DEM. Let the given proportions be

$$a : b :: c : d,$$

and $m : n :: p : q.$

These are the same as $\frac{a}{b} = \frac{c}{d}, \qquad (1)$

and $\frac{m}{n} = \frac{p}{q}. \qquad (2)$

Multiplying (1) by (2), $\frac{am}{bm} = \frac{cp}{dq};$

that is, $am : bn :: cp : dq.$

From the given proportions,

$$ad = bc, \tag{3}$$

$$mq = np. \tag{4}$$

Dividing (3) by (4), $\frac{ad}{mq} = \frac{bc}{np},$

or $$\frac{a}{m} \times \frac{d}{q} = \frac{b}{n} \times \frac{c}{p}.$$

Hence, by Art. 297, $\frac{a}{m} : \frac{b}{n} :: \frac{c}{p} : \frac{d}{q}.$

302. **Theorem.** *In a series of equal ratios the sum of all the antecedents is to the sum of all the consequents as any antecedent is to its consequent.*

Dem. If $a : b :: c : d :: e : f :: g : h$, etc., (1)

then $a + c + e + g +$ etc. $: b + d + f + h +$ etc. $:: a : b$ or $c : d$, etc.

We have the identity, $ab = ba.$

From (1) we have $ad = bc,$

$$af = be,$$

$$ah = bg,$$

etc., etc.

Adding, $a(b + d + f + h + \text{etc.}) = b(a + c + e + g + \text{etc.}).$

Hence, by Art. 297,

$$a + c + e + g + \text{etc.} : b + d + f + h + \text{etc.} :: a : b.$$

303. When a proportion is given and we wish to determine whether some other proposed relation involving the same quantities is true, we may proceed in any one of several ways. For example, let the given proportion be

$$a : b :: c : d,$$

and let the problem be to determine whether

$$a + b : b :: c + d : d.$$

1st. We may proceed as in the demonstration of this case, Art. 198, c.

2d. The proposed proportion is true if it can be shown that the product of the extremes equals the product of the means. This would give

$$ad + bd = bc + bd,$$

or

$$ad = bc,$$

which is seen from the given proportion to be true.

3d. In the given proportion we may represent each of the equal ratios, $a : b$ and $c : d$, by r, giving

$$\frac{a}{b} = r \text{ and } \frac{c}{d} = r,$$

or

$$a = br \text{ and } c = dr.$$

Substituting these values in the proposed form, we have

$$br + b : b :: dr + d : d,$$

in which the ratios are seen to be equal, each being $r + 1 : 1$.

Any of the forms of the preceding theorems may be tested in this way.

EXAMPLES LXXVII

1. Find the ratio of x to y in $7x - 5y : 4x - 3y :: 5 : 2$.

2. From $a : b :: c : d$ deduce

$$5a + 3b : 5a - 3b :: 5c + 3d : 5c - 3d.$$

3. From $x : y :: 4 : 7$, find the ratio of $x - 4$ to $y - 7$.

4. Find the mean proportional between $x^2 - \frac{1}{y^2}$ and $y^2 - \frac{1}{x^2}$.

5. Find x from $x^2+5x+6 : x^2+10x+21 :: x^2-4 : 4x^2+8x-32$.

6. The third proportional to two numbers is 48, and the mean proportional between them is 6. Find the numbers.

7. If a, b, c, d are in continued proportion, prove that $b + c$ is a mean proportional between $a + b$ and $c + d$.

8. From $a : b :: c : d$ deduce $a - c : b - d :: \sqrt{a^2 + c^2} : \sqrt{b^2 + d^2}$.

9. From $\frac{1}{2}a - x : \frac{1}{2}a + x :: b - y : b + y$ deduce $2x : y :: a : b$.

Sug. Take by division and composition, divide second couplet by 2, and take by alternation.

10. If $a : b :: c : d :: e : f$, prove that

$$a + 3c + 2e : a - e :: b + 3d + 2f : b - f.$$

11. What operations on $x : y :: a : b$ will produce

$$x^2 : a^2 :: x^2 + y^2 : a^2 + b^2?$$

12. If $a : b :: p : q$, prove that $a^2 + b^2 : \dfrac{a^3}{a+b} :: p^2 + q^2 : \dfrac{p^3}{p+q}$.

13. What quantity added to each of the quantities a, b, c, d, will make them proportionals?

14. If four quantities are proportionals, show that there is no quantity which, being added to each, will leave the sums proportionals.

15. If the fourth proportional to a, b, c is the same as that to a, b', c', show that $b : b' :: c' : c$.

16. If $a : b :: c : d$, prove that $a^2 + b^2 : c^2 + d^2 :: (a + b)^2 : (c + d)^2$.

17. If $a : b :: c : d$, prove that $ab + cd$ is a mean proportional between $a^2 + c^2$ and $b^2 + d^2$.

18. What quantity subtracted from each of the quantities a, b, c will leave the remainders in continued proportion?

19. If x be to y in the duplicate ratio of a to b, and a be to b in the sub-duplicate ratio of $a + x$ to $a - y$, prove that

$$2x : a :: x - y : y.$$

20. A farmer's crop of wheat was to his crop of oats as $2 : 3$. His neighbor raised 50 bushels more of each, and his crop of wheat was to his crop of oats as $5 : 7$. How many bushels of each did the first farmer raise?

Sug. If in solving such problems proportions are used, place the product of the extremes in each equal to the product of the means, and solve the equations in the usual way. In this example but one unknown quantity is necessary, as we may represent by $2x$ and $3x$ the number of bushels of wheat and oats respectively.

21. Divide 14 into two such parts that the greater divided by the less shall be to the less divided by the greater as 16 to 9.

22. Two vessels contain respectively 15 and $27\frac{1}{2}$ gallons. How many gallons must be transferred from one to the other that the amounts may be in the ratio $2:3$?

23. It is required to find a number such that the sum of its digits is to the number itself as 4 to 13, and the difference of its digits is to the number expressed when the digits are interchanged as 2 to 31.

24. Two numbers having the same two digits are to each other as $5:6$. What are the numbers?

25. Find two numbers such that their sum, difference, and product may be as the numbers s, d, and p, respectively.

26. Find two numbers whose difference is to the difference of their squares as $m:n$, and whose sum is to the difference of their squares as $a:b$.

27. A bin contains chop-feed composed of corn and oats. A second bin contains 6 bushels more of each, and has 7 bushels of corn to every 6 bushels of oats. A third bin contains 6 bushels less of each than the first, and has 6 bushels of corn to every 5 bushels of oats. How many bushels of each does the first bin contain?

28. The force of the earth's attraction is inversely as the square of the distance from the center. At the surface this force is expressed by the number 32.16. By what is it expressed at the moon, whose distance from the center of the earth is 60 radii of the earth?

29. The velocities of bodies revolving around another body are inversely proportional to the squares of the distances. If the velocity is v when the distance is r, what is it when the distance is r'?

30. Two men have equal capital in business, and one is losing as fast as the other is gaining. A third man's capital is less by p dollars, but is rapidly increasing. Show that if m and n are the respective gains of the third man at the times when he has the same amounts as the others, then $2mn = p(m+n)$.

31. Before noon, a clock which is too fast and points to afternoon time, is turned back 5 hours and 40 minutes to the true time; and it is observed that the time before shown is to the true time as 29 to 105. Find the true time.

32. Areas of circles are as the squares of their diameters. Two circular metallic plates, each an inch thick, whose diameters are 6 and 8 inches respectively, are melted and cast into a single circular plate, 1 inch thick. Find its diameter.

33. The volumes of spheres are as the cubes of their radii. Find the radius of a sphere whose volume is equal to the sum of the volumes of three spheres whose radii are 3, 4, and 5, respectively.

34. The rates of an express train and an accommodation train differ by 15 miles an hour, and their times of making a distance of 180 miles are as 9 : 14. The express train loses by stoppages only half as much time as the accommodation train, and the latter thus loses as much time as it would require in running 30 miles. Find the rates.

35. Two passengers have together 560 pounds of baggage, and are charged for the excess above the weight allowed 62 cents and \$1.18 respectively. A third passenger has as much baggage as the other two, and is charged \$2.30 for excess. How much baggage is each passenger allowed without charge?

SECTION III — VARIATION

304. When the ratio of two quantities is constant, either is said to *vary directly* as the other, or simply to *vary* as the other.

Thus, if

$$x = my,$$

in which m is constant, any change in y causes x to change in the same ratio: if y be doubled, x will be doubled; if y be tripled, x will be tripled; if y be halved, x will be halved; and so on.

This relation between x and y is written

$$x \propto y,$$

and is read "x varies as y."

Distance traveled at a given rate varies as the time, the constant ratio being the given rate. If d, r, and t represent distance, rate, and time respectively,

$$d = rt,$$

and

$$d \propto t.$$

Amount earned in a given time varies as the daily wages, the constant ratio being the given time. If a, t, and w represent amount, time, and daily wages respectively,

$$a = tw,$$

and

$$a \propto w.$$

The circumference of a circle varies as its radius, the constant ratio being 2π. If c and r represent the circumference and radius respectively,

$$c = 2\pi r,$$

and

$$c \propto r.$$

305. When the ratio of a quantity to the reciprocal of another is constant, either is said to *vary inversely* as the other.

Thus, if

$$x = \frac{m}{y},$$

in which m is constant, any change in y causes x to change in the inverse ratio (Art. 285): if y be doubled, x will be halved; if y be tripled, x will be made one third as great; if y be halved, x will be doubled; and so on.

This relation between x and y is written

$$x \propto \frac{1}{y},$$

and is read, "x varies inversely as y."

The time required to travel a given distance varies inversely as the rate, the constant ratio being the given distance. If t, d, and r represent time, distance, and rate respectively,

$$t = \frac{d}{r},$$

and

$$t \propto \frac{1}{r}.$$

The time required to earn a given amount varies inversely as the daily wages, the constant ratio being the given amount. If t, d, and w represent time, amount, and daily wages respectively,

$$t = \frac{a}{w},$$

and

$$t \propto \frac{1}{w}.$$

306. When the ratio of one quantity to the product of two or more other quantities is constant, the first is said to *vary jointly* as the others.

Thus, if

$$x = myz,$$

in which m is constant, any change in y and z causes x to change in a ratio represented by the product of these changes; if y be doubled and z tripled, x will be made six times as great; and so on.

This relation between x and yz is written

$$x \propto yz,$$

and is read, "x varies jointly as y and z."

Distance traveled varies jointly as the time and rate, the constant ratio being 1.

The area of a triangle varies jointly as its base and altitude, the constant ratio being $\frac{1}{2}$. If s, b, and h represent area, base, and altitude respectively,

$$s = \tfrac{1}{2} bh,$$

and

$$s \propto bh.$$

307. When the ratio of one quantity to the quotient of a second divided by a third is constant, the first is said to *vary directly* as the second and *inversely* as the third.

Thus, if

$$x = m\frac{y}{z},$$

in which m is constant, then

$$x \propto \frac{y}{z},$$

which is read, "x varies directly as y and inversely as z."

If time, distance, and rate are all variable, the time varies directly as the distance and inversely as the rate.

In doing work, the time varies directly as the amount of work and inversely as the number of workmen employed.

308. **Theorem.** *If one set of corresponding values of the variables of a variation be given, the constant ratio becomes known.*

Dem. Let the variation be

$$x \propto y. \qquad (1)$$

By definition (Art. 304) the ratio of x to y is constant. Let this constant ratio be m. Then

$$x = my. \qquad (2)$$

Or we may say, since x varies as y, x is always a certain number of times y, as m times y.

Let a and b be corresponding values of x and y. Substituting these in (2), we have

$$a = mb,$$

whence $$m = \frac{a}{b}.$$

The same reasoning applies to the other forms of variation.

309. Cor. *To pass from a variation to an equation, a constant factor, the ratio, is introduced; and,* conversely, *to pass from an equation to a variation, constant factors are omitted.*

310. Caution. When two or more variations occur in the same problem, the same constant must not be used twice in passing from variations to equations; for, while in each variation the ratio is constant, it will not do to assume that it is the same.

311. Theorem. *A variation may always be expressed as a proportion, and is, in fact, simply a contracted proportion.*

Dem. By definition the expression

$$x \propto y$$

signifies that whatever value x may have, y has a corresponding value, such that the ratio of x to y is constant. Letting x' and x'' be two values of x, and y' and y'' the corresponding values of y, the ratio $x' : y'$ is the same as the ratio $x'' : y''$; that is,

$$x' : y' :: x'' : y'',$$

or, if we choose (Art. 298, b),

$$x' : x'' :: y' : y''.$$

The same reasoning applies to the other forms of variation.

EXAMPLES LXXVIII

1. If $x \propto y$ and $y \propto \frac{1}{z}$, show how x varies with reference to z.

Solution. Since $$x \propto y,$$
we have (Art. 309) $$x = my; \qquad (1)$$
and since $$y \propto \frac{1}{z},$$
we have $$y = \frac{n}{z}. \qquad (2)$$

Eliminating y from (1) and (2),

$$x = \frac{mn}{z},$$

whence (Art. 309) $$x \propto \frac{1}{z}.$$

2. If $x \propto z$ and $y \propto z$, show how x varies with reference to y.

3. If $x \propto y^2$ and $y \propto z^2$, show how x varies with reference to z.

4. If $x^2 \propto z$ and $y^2 \propto z$, prove that $xy \propto z$.

5. If $x \propto y$ and $x = 14$ when $y = \frac{2}{5}$, what is the value of x in terms of y?

Solution. Since $$x \propto y,$$

we have (Art. 309) $$x = my.$$

Substituting the given corresponding values,

$$14 = \tfrac{2}{5}\,m,$$

whence $$m = 35.$$

Since m is constant, when we have found its value for one pair of corresponding values of x and y, we have it for all corresponding values of x and y. Hence

$$x = 35\,y.$$

6. If $x \propto y$, and $x = 20$ when $y = 2\frac{1}{2}$, what is the value of x in terms of y?

7. If $x \propto \frac{1}{y}$, and $x = 13$ when $y = 4$, what is the value of x in terms of y?

8. If $x \propto \frac{1}{y^2}$, and $x = 8$ when $y = \frac{1}{2}$, what is the value of x in terms of y?

9. If $x \propto y$, and $x = 12$ when $y = 3$, what is the value of y when $x = 32$?

1st Solution. Since $$x \propto y, \qquad (1)$$

we have (Art. 309) $$x = my. \qquad (2)$$

Substituting the first pair of values,

$$12 = 3\,m,$$

whence $$m = 4.$$

Substituting this and the second given value of x in (2),

$$32 = 4\,y,$$

whence $$y = 8.$$

2D SOLUTION. Since $x \propto y,$

we have (Art. 311) $x' : y' :: x'' : y''.$

Substituting the given values,

$$12 : 3 :: 32 : y'',$$

whence $y'' = 8.$

CAUTION. In solving such examples, students sometimes substitute given values directly in the variation, thus: "$12 \propto 3$, etc." As 12 and 3 cannot vary, it is absurd to write $\propto$ between them.

10. If $x \propto y$, and $x = 15$ when $y = 2\frac{1}{2}$, what is the value of x when $y = 4$?

11. If $x \propto \frac{1}{y}$, and $x = 10$ when $y = 3$, what is the value of y when $x = 6$?

12. If $x \propto \frac{1}{y^2}$, and $x = 4$ when $y = 3$, what is the value of x when $y = 2$?

13. If $x \propto yz$, and $x = 24$ when $y = 2$ and $z = 3$, what is the value of y when $x = 8$ and $z = 4$?

14. If $x \propto \frac{y}{z}$, and $x = 12$ when $y = 8$ and $z = 2$, what is the value of z when $x = 9$ and $y = 12$?

15. If x varies directly as y and inversely as the square of z, and $x = 6$ when $y = 8$ and $z = 2$, find the value of x when $y = 18$ and $z = 3$.

16. The volume of a sphere varies as the cube of its radius. If the volume of a soap bubble is 5.888 when its radius is 1 inch, what is its volume when its radius is $1\frac{1}{2}$ inches?

SOLUTION. Since $v \propto r^3,$

we have (Art. 309) $v = mr^3.$

Substituting the given values of v and r,

$$m = 5.888.$$

Substituting this value of m and the second value of r,

$$v = 5.888\left(\tfrac{3}{2}\right)^3 = 19.872.$$

Or, if we choose, we may use a proposition, thus:

$$v : v' :: r^3 : r'^3,$$

$$5.888 : v' :: 1 : \tfrac{27}{8},$$

whence $$v' = 19.872.$$

17. If a metal ball whose radius is 2 inches weighs 6 pounds, what is the weight of a ball of the same metal whose radius is 4 inches?

18. The distance fallen by a body from rest varies as the square of the time of falling. If a body falls $257\frac{1}{3}$ feet in 4 seconds, how far will it fall in 6 seconds?

19. Amount of illumination varies directly as the intensity of the light and inversely as the square of the distance from the light. What must be the intensity of a light to give at the distance of 75 feet 3 times the illumination of one whose intensity is 10 and distance 50 feet?

20. The volume of a pyramid varies jointly as its base and altitude. A pyramid whose base is 9 feet square and whose height is 10 feet contains 10 cubic yards. What must be the height of a pyramid with a base 3 feet square in order that it may contain 2 cubic yards?

21. The volume of a right circular cone varies jointly as its height and the square of the radius of its base. If the volume of a certain cone is 94 cubic inches, what is the volume of another cone twice as high and the radius of whose base is half as great?

22. The volume of a gas varies as the absolute temperature and inversely as the pressure. When the temperature in a given case is 260 and the pressure 15, the volume is 200 cubic inches. What will be the volume when the temperature becomes 390 and the pressure 18?

23. The pressure of the wind on a plane area varies jointly as the area and the square of the velocity of the wind. If the pressure on 1 square foot is 1 pound when the velocity of the wind is 16 miles an hour, what is the velocity of the wind when the pressure on 2 square yards is 50 pounds?

24. If $x + y \propto x - y$, prove that $x^2 + y^2 \propto xy$.

25. Given $y = p + q$, in which $p \propto x$, and $q \propto \frac{1}{x}\cdot$ When $x = 1$, $y = 6$; and when $x = 2$, $y = 5$. Find the value of y in terms of x.

26. Given $x = a + p + q$, in which a is constant, $p \propto y$, and $q \propto y^2$. When $x = 6, 17, 34$, $y = 1, 2, 3$, respectively. Find the value of x in terms of y.

27. Given that $s \propto t^2$ when f is constant, and $s \propto f$ when t is constant; also $2s = f$ when $t = 1$. Find the value of s in terms of f and t.

28. By Kepler's third law the square of a planet's period of revolution around the sun varies as the cube of its mean distance from the sun. If r is the distance of a planet whose period is t, what is the distance of a planet whose period is t'?

29. Attraction varies directly as the mass of the attracting body and inversely as the square of the distance from its center. The mass of the sun being 332,000 times that of the earth, and its radius 109.4 times that of the earth, what is the weight of a body at the surface of the sun as compared with its weight at the surface of the earth?

30. The mass of the sun being 1047.35 times that of Jupiter, and the distance of Jupiter from the sun being 414 times the distance from Jupiter to his outer satellite, what is the attraction of the sun on Jupiter as compared with Jupiter's attraction on his outer satellite?

CHAPTER XIV

PROGRESSIONS

SECTION I — ARITHMETICAL PROGRESSION

312. **A Series** is a succession of terms proceeding according to a definite law.

313. A series is **Increasing** or **Decreasing**, or **Ascending** or **Descending**, according as the terms increase or decrease.

314. **An Arithmetical Progression** is a series in which each term is greater or less than the preceding term by a constant quantity, called the **Common Difference.**

If we regard each term as being obtained by the addition of the common difference to the preceding term, the common difference is plus in an increasing, and minus in a decreasing, arithmetical progression.

Thus, 1, 3, 5, 7, etc., is an increasing arithmetical progression, in which the common difference is 2.

15, 10, 5, 0, -5, -10, etc., is a decreasing arithmetical progression, in which the common difference is -5.

a, $a+d$, $a+2d$, $a+3d$, etc., is the general form of an arithmetical progression, the common difference being d, which may be either plus or minus.

315. In the treatment of arithmetical progression, five elements are involved:

1. The first term, a.
2. The last term, l.
3. The common difference, d.
4. The number of terms, n.
5. The sum of the terms, S.

316. Arithmetical Means between any two quantities are the terms that lie between them in an arithmetical progression.

When there is but one intermediate term, it is called the **Arithmetical Mean** of (or between) the other two.

317. Theorem. *The arithmetical mean of two quantities is equal to half their sum.*

DEM. Let a, b, c be in arithmetical progression.

By definition $$b - a = c - b,$$

whence $$b = \tfrac{1}{2}(a + c).$$

318. Theorem. *The formula for the last, or nth, term of an arithmetical progression in terms of the first term, the common difference, and the number of terms, is*

$$l = a + (n - 1)\, d.$$

DEM. In the general form of an arithmetical progression,

$$a,\ a + d,\ a + 2\,d,\ a + 3\,d,\ a + 4\,d,\ \text{etc.},$$

it is seen that each term is formed by adding to a the product of d and a number which is 1 less than the number of the term; that is,

$$l = a + (n - 1)d.$$

319. Theorem. *The formula for the sum of an arithmetical progression in terms of the first term, the last term, and the number of terms, is*

$$S = \left(\frac{a + l}{2}\right)n.$$

DEM. Taking the sum of the terms of the general form of an arithmetical progression and the sum of the same terms in the reverse order, we have

$$S = a + (a + d) + (a + 2\,d) + \cdots (l - 2\,d) + (l - d) + l, \quad (1)$$

and $$S = l + (l - d) + (l - 2\,d) + \cdots (a + 2\,d) + (a + d) + a. \quad (2)$$

Adding (1) and (2),

$$2\,S = (a + l) + (a + l) + (a + l) + \cdots (a + l) + (a + l) + (a + l).$$

It is seen that $(a+l)$ is taken as many times as there are terms. Hence

$$2S = (a+l)n,$$

or

$$S = \left(\frac{a+l}{2}\right)n.$$

320. Any one of the four quantities involved in either of the equations,

$$l = a + (n-1)d,$$

$$S = \left(\frac{a+l}{2}\right)n,$$

may be found in terms of the other three; and by combining the two equations, any one of the five quantities involved may be eliminated, and any one of the remaining four found in terms of the other three. The twenty formulæ on page 198 result. They are convenient, but not necessary, as all cases may be solved by the two fundamental formulæ, either directly or by first finding an intermediate element. The student should reserve, until after he has taken the subject of Quadratic Equations, the development of numbers 2, 11, 18, and 20.

EXAMPLES LXXIX

1. Find the 20th term and the sum of 20 terms of 1, 4, 7, 10, etc.

2. Find the 21st term and the sum of 21 terms of 3, 7, 11, 15, etc.

3. Find the 36th term and the sum of 36 terms of 12, 10, 8, 6, etc.

4. Find the 10th term and the sum of 10 terms of 3, $2\frac{1}{3}$, $1\frac{2}{3}$, etc.

In each of the following, find the two elements not given:

5. $a=1,\ d=5,\ n=24.$

6. $l=71,\ d=5,\ n=15.$

7. $a=13,\ l=73,\ n=11.$

8. $a=10,\ l=87,\ d=7.$

9. $d=4,\ n=14,\ s=812.$

10. $a=-5,\ n=19,\ s=-950.$

11. $a=7,\ l=143,\ s=1350.$

12. $l=-\frac{3}{5},\ n=19,\ s=0.$

Formulæ in Arithmetical Progression

Number	Given	Required	Formulæ
1.	$a,\ d,\ n$	l	$l = a + (n-1)d,$
2.	$a,\ d,\ S$	l	$l = -\frac{1}{2}d \pm \sqrt{\{2dS + (a - \frac{1}{2}d)^2\}},$
3.	$a,\ n,\ S$	l	$l = \frac{2S}{n} - a,$
4.	$d,\ n,\ S$	l	$l = \frac{S}{n} + \frac{(n-1)d}{2}.$
5.	$a,\ d,\ n$	S	$S = \frac{1}{2}n\{2a + (n-1)d\},$
6.	$a,\ d,\ l$	S	$S = \frac{l+a}{2} + \frac{l^2 - a^2}{2d},$
7.	$a,\ n,\ l$	S	$S = (l+a)\frac{n}{2},$
8.	$d,\ n,\ l$	S	$S = \frac{1}{2}n\{2l - (n-1)d\}.$
9.	$d,\ n,\ l$	a	$a = l - (n-1)d,$
10.	$d,\ n,\ S$	a	$a = \frac{S}{n} - \frac{(n-1)d}{2},$
11.	$d,\ l,\ S$	a	$a = \frac{1}{2}d \pm \sqrt{(l + \frac{1}{2}d)^2 - 2dS},$
12.	$n,\ l,\ S$	a	$a = \frac{2S}{n} - l.$
13.	$a,\ n,\ l$	d	$d = \frac{l-a}{n-1},$
14.	$a,\ n,\ S$	d	$d = \frac{2(S-an)}{n(n-1)},$
15.	$a,\ l,\ S$	d	$d = \frac{l^2 - a^2}{2S - l - a},$
16.	$n,\ l,\ S$	d	$d = \frac{2(nl - S)}{n(n-1)}.$
17.	$a,\ d,\ l$	n	$n = \frac{l-a}{d} + 1,$
18.	$a,\ d,\ S$	n	$n = \frac{\pm\sqrt{(2a-d)^2 + 8dS} - 2a + d}{2d},$
19.	$a,\ l,\ S$	n	$n = \frac{2S}{l+a},$
20.	$d,\ l,\ S$	n	$n = \frac{2l + d \pm \sqrt{(2l+d)^2 - 8dS}}{2d}.$

13. Insert 3 arithmetic means between 73 and 193.

SUG. First find d, n being 5.

14. Insert 11 arithmetic means between -49 and 59.

15. Find the first term of an arithmetical progression whose 59th term is $-2\frac{1}{2}$, and 60th term $-1\frac{3}{4}$.

16. Find the first term of an arithmetical progression whose 2d term is $\frac{1}{2}$, and 55th term 5.8.

17. Find the sum of 24 terms of an arithmetical progression whose 13th term is 25, and 19th term 37.

18. Find the sum of 12 terms of the arithmetical progression whose 1st term is 38, and 4th term 86.

19. Between what two terms of 3, 8, 13, etc., does 391 lie?

20. If a body falling from rest descends a feet the first second, $3a$ the second, $5a$ the third, and so on, how far will it fall during the tth second, and how far in t seconds?

21. If a body falling from rest descends $16\frac{1}{12}$ feet the first second, $48\frac{1}{4}$ feet the second, $80\frac{5}{12}$ feet the third, and so on, how far will it fall in 30 seconds?

22. A man training for a mile race ran over the course every day for 21 days. His time the first day was 8 minutes, but this was gradually diminished, his whole time for the 21 runs being 133 minutes. What was the average daily diminution of time, and what was his time the last day?

23. Some fence posts are to be carried from a pile to the holes where they are to be set. How far must a laborer walk, carrying one post at a time, to place them in the 50 post holes, which are in a straight line and 8 feet apart, the first one being at the pile?

24. A man bought an estate which yielded \$1500 profit the first year. His personal expenses for the first year were \$1250. His income from the estate increased \$100 yearly, and his personal expenses increased \$125 yearly. After how many years were his personal expenses equal to the income from his estate?

SECTION II.—GEOMETRICAL PROGRESSION

321. A Geometrical Progression is a series in which each term is equal to the preceding term multiplied by a constant quantity, called the **Ratio.**

The ratio is greater than unity in an increasing and less than unity in a decreasing geometrical progression.

Thus, 2, 4, 8, 16, etc., is an increasing geometrical progression in which the ratio is 2.

12, 4, $\frac{4}{3}$, $\frac{4}{9}$, $\frac{4}{27}$, etc., is a decreasing geometrical progression in which the ratio is $\frac{1}{3}$.

a, ar, ar^2, ar^3, ar^4, etc., is the general form of a geometrical progression, the ratio being r.

322. In the treatment of geometrical progression, five elements are involved:

1. The first term, a.
2. The last term, l.
3. The ratio, r.
4. The number of terms, n.
5. The sum of the terms, S.

323. Geometric Means between any two quantities are the terms that lie between them in a geometrical progression.

When there is but one intermediate term, it is called the **Geometric Mean** of (or between) the other two.

324. Theorem. *The geometric mean of two quantities is equal to the square root of their product.*

DEM. Let a, b, c, be in geometric progression.

By definition $$\frac{b}{a}=\frac{c}{b},$$

whence $$b=\sqrt{ac}.$$

325. SCH. The geometric mean of two quantities is the same as the mean proportional between them (Arts. 287 and 295).

326. Theorem. *The formula for the last, or nth, term of a geometrical progression in terms of the first term, the ratio, and the number of terms, is*

$$l = ar^{n-1}.$$

DEM. In the general form of a geometrical progression,

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4, \text{ etc.},$$

it is seen that each term is formed by multiplying a by r affected with an exponent 1 less than the number of the term; that is,

$$l = ar^{n-1}.$$

327. Theorem. *The formula for the sum of a geometrical progression in terms of the first term, the ratio, and the number of terms, is*

$$S = \frac{ar^n - a}{r - 1}.$$

DEM. Taking the sum of the terms of the general form of a geometrical progression and multiplying this series by r, we have

$$S = a + ar + ar^2 + \cdots ar^{n-3} + ar^{n-2} + ar^{n-1}, \qquad (1)$$

and $$rS = \phantom{a +{}} ar + ar^2 + \cdots ar^{n-3} + ar^{n-2} + ar^{n-1} + ar^n. \qquad (2)$$

Subtracting (1) from (2),

$$rS - S = ar^n - a,$$

whence $$S = \frac{ar^n - a}{r - 1}.$$

328. Theorem. *The formula for the sum of a geometrical progression in terms of the first term, the last term, and the ratio, is*

$$S = \frac{lr - a}{r - 1}.$$

DEM. From $$l = ar^{n-1}$$

we have $$lr = ar^n.$$

Substituting this value of ar^n in the formula for S, Art. 327,

$$S = \frac{lr - a}{r - 1}.$$

Formulæ in Geometrical Progression

Number	Given	Required	Formulæ
1.	$a,\ r,\ n$	l	$l = ar^{n-1}$,
2.	$a,\ r,\ S$		$l = \dfrac{a + (r-1)S}{r}$,
3.	$a,\ n,\ S$		$l(S-l)^{n-1} - a(S-a)^{n-1} = 0$,
4.	$r,\ n,\ S$		$l = \dfrac{(r-1)Sr^{n-1}}{r^n - 1}$.
5.	$a,\ r,\ n$	S	$S = \dfrac{a(r^n-1)}{r-1}$,
6.	$a,\ r,\ l$		$S = \dfrac{rl-a}{r-1}$,
7.	$a,\ n,\ l$		$S = \dfrac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}$,
8.	$r,\ n,\ l$		$S = \dfrac{lr^n - l}{r^n - r^{n-1}}$.
9.	$r,\ n,\ l$	a	$a = \dfrac{l}{r^{n-1}}$,
10.	$r,\ n,\ S$		$a = \dfrac{(r-1)S}{r^n-1}$,
11.	$r,\ l,\ S$		$a = rl - (r-1)S$,
12.	$n,\ l,\ S$		$a(S-a)^{n-1} - l(S-l)^{n-1} = 0$.
13.	$a,\ n,\ l$	r	$r = \sqrt[n-1]{\dfrac{l}{a}}$,
14.	$a,\ n,\ S$		$r^n - \dfrac{S}{a}r + \dfrac{S-a}{a} = 0$,
15.	$a,\ l,\ S$		$r = \dfrac{S-a}{S-l}$,
16.	$n,\ l,\ S$		$r^n - \dfrac{S}{S-l}r^{n-1} + \dfrac{l}{S-l} = 0$.
17.	$a,\ r,\ l$	n	$n = \dfrac{\log l - \log a}{\log r} + 1$,
18.	$a,\ r,\ S$		$n = \dfrac{\log[a + (r-1)S] - \log a}{\log r}$,
19.	$a,\ l,\ S$		$n = \dfrac{\log l - \log a}{\log(S-a) - \log(S-l)} + 1$,
20.	$r,\ l,\ S$		$n = \dfrac{\log l - \log[lr - (r-1)S]}{\log r} + 1$.

329. Theorem. *The formula for the sum of an infinite, decreasing geometrical progression is*

$$S=\frac{a}{1-r}.$$

Dem. Since in a decreasing geometrical progression r is less than unity, the term ar^n, in the formula

$$S=\frac{ar^n-a}{r-1},$$

has no appreciable value when n is infinite. Hence this formula becomes

$$S=\frac{-a}{r-1}=\frac{a}{1-r}.$$

330. Any one of the four quantities involved in any one of the equations,

$$l=ar^{n-1},$$

$$S=\frac{ar^n-a}{r-1},$$

$$S=\frac{lr-a}{r-1},$$

may be found in terms of the other three; and by combining the first with one of the others, any one of the five quantities involved may be eliminated, and any one of the remaining four found in terms of the other three, except in a few cases involving higher literal equations. These last can be solved when numerical quantities are substituted for the literal. The twenty formulæ on page 202 result. The student should reserve until after taking the subject of Logarithms the development of the last four.

EXAMPLES LXXX

1. Prove the following properties of a geometrical progression:

(*a*) The alternate terms, or any terms separated by the same intervals, are in geometrical progression.

(*b*) The products of the terms by the same quantity are in geometrical progression.

(*c*) The same powers of the terms are in geometrical progression.

(*d*) The reciprocals of the terms are in geometrical progression.

(*e*) Any term is a mean proportional between any two terms separated from it by the same intervals.

(*f*) The product of any odd number, p, of consecutive terms is equal to the pth power of the middle one.

(*g*) The product of any two terms is equal to the product of any other two terms separated from these by the same intervals in opposite directions.

2. Find the 12th term and the sum of 12 terms of 1, 2, 4, 8, etc.

3. Find the 10th term and the sum of 10 terms of 1, 3, 9, 27, etc.

4. Find the 11th term and the sum of 11 terms of $\frac{1}{64}$, $\frac{1}{16}$, $\frac{1}{4}$, 1, etc.

5. Find the 11th term and the sum of 11 terms of 3, -6, 12, -24, etc.

In each of the following find the two elements not given:

6. $r=2,\quad n=8,\quad S=1275.$ **7.** $r=3,\quad n=9,\quad l=26{,}244.$

8. $a=-\frac{2}{3}, n=7,\quad r=-\frac{1}{2}.$ **9.** $l=256, n=9,\quad r=2.$

10. $a=-2, n=6,\quad l=2048.$ **11.** $a=2,\quad n=7,\quad l=145.$

12. $a=1,\quad l=81,\quad r=3.$ **13.** $l=160,\quad r=2,\quad S=315.$

14. Insert 3 geometric means between 17 and 4352.

Sug. First find r, n being 5.

15. Insert 4 geometric means between 26 and 6318.

16. Find the 1st term of a geometrical progression whose 5th term is 336, and 9th term 5376.

17. Find the sum of 8 terms of the geometrical progression whose 4th term is 108, and 7th term 2916.

18. Find the 11th term of a geometrical progression whose 7th term is 192, and 10th term -1536.

19. The sum of the first 8 terms of a geometrical progression is 17 times the sum of the first 4 terms. Find the ratio.

20. The 1st term of a geometrical progression is 3, and the sum of the first 3 terms is one eighth of the sum of the next 3 terms. Find the ratio.

21. The sum of the 1st and 2d terms of a geometrical progression is 30, and the sum of the 4th and 5th is 1920. Find the first 5 terms of the progression.

22. Find the sum of 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc., to infinity.

23. Find the value of $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} +$ etc., to infinity.

24. Find the value of $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} +$ etc., to infinity.

25. Find the value of $.4\dot{2}\dot{3}$.

Sug. $.4\dot{2}\dot{3} = .4232323\ldots = .4 + .023 + .00023 +$ etc.

26. Find the value of $.2\dot{7}$.

27. Find the value of $.\dot{3}1\dot{2}$.

28. Each term in a certain infinite decreasing geometrical progression is equal to the sum of all that follow it. Find the ratio.

29. What is the distance passed through before coming to rest by a ball which falls from a height of 50 feet and at every fall rebounds half the distance?

30. A "letter chain" is started for the benefit of a public charity, three letters, each numbered 1, being sent out by the starter with the request that each of the recipients remit 10 cents and send out three other letters, each numbered 2, with a similar request, and so on, until the numbers reach 25. Should all comply, (*a*) How much would be realized for the charity? (*b*) What would be the entire outlay for postage at 2 cents for each letter? (*c*) With a uniform distribution, how many times would each of the 75 million inhabitants of the United States respond?

SECTION III. — HARMONIC PROGRESSION

331. A Harmonic Progression is a series of terms whose reciprocals are in arithmetical progression.

Thus, 1, $\frac{1}{4}$, $\frac{1}{7}$, $\frac{1}{10}$, $\frac{1}{13}$, etc., are in harmonic progression, because their reciprocals, 1, 4, 7, 10, 13, etc., are in arithmetical progression.

The general form of a harmonic progression is $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}$, etc., in which a is the first term of an arithmetical progression, and d the common difference.

Most problems concerning quantities in harmonic progression are solved by treating the arithmetical progression obtained by taking the reciprocals of the quantities that are in harmonic progression.

332. If the lengths of strings of the same substance, size, and tension be proportional to the terms of a harmonic progression, any two of these strings vibrating together produce harmony of sound; hence the term *harmonic.*

333. Harmonic Means between any two quantities are the terms that lie between those quantities in a harmonic progression.

When there is but one intermediate term, it is called the **Harmonic Mean** of (or between) the other two.

334. Theorem. *The harmonic mean of two quantities is twice their product divided by their sum.*

Dem. If a, b, c are in harmonic progression, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are, by definition, in arithmetical progression; hence

$$\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b},$$

whence
$$b=\frac{2ac}{a+c}.$$

335. Theorem. *If three quantities are in harmonic progression, the difference between the first and second is to the difference between the second and third as the first is to the third.*

Dem. If a, b, c are in harmonic progression, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are, by definition, in arithmetical progression; hence

$$\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}.$$

Clearing of fractions, $ac - bc = ab - ac$,

or
$$c(a-b)=a(b-c);$$

therefore (Art. 297) $\quad a-b : b-c :: a : c.$

336. Theorem. *The formula for the last, or nth, term of a harmonic progression in terms of the first term, the second term, and the number of terms, is*

$$l=\frac{ab}{b+(n-1)(a-b)}.$$

Dem. If a, b, etc., are in harmonic progression, $\frac{1}{a}$, $\frac{1}{b}$, etc., are, by definition, in arithmetical progression; hence

$$d=\frac{1}{b}-\frac{1}{a}=\frac{a-b}{ab}.$$

Substituting in $l'=a'+(n-1)d$ (Art. 318),

$$\frac{1}{l}=\frac{1}{a}+(n-1)\frac{a-b}{ab}=\frac{b+(n-1)(a-b)}{ab},$$

whence $$l=\frac{ab}{b+(n-1)(a-b)}.$$

337. Note. No formula for the sum of a harmonic progression is known.

EXAMPLES LXXXI

1. Find the 23d term of $\frac{1}{3}, \frac{1}{8}, \frac{1}{13}, \frac{1}{18}$, etc.

2. Find the 4th and 7th terms of a harmonic progression whose 2d term is $\frac{1}{7}$, and 5th term $\frac{1}{16}$.

3. Find the harmonic mean of 21 and 42.

4. Insert 3 harmonic means between 14 and 42.

Sug. Find 3 arithmetic means between $\frac{1}{14}$ and $\frac{1}{42}$, and take their reciprocals.

5. Insert 3 harmonic means between $\frac{1}{23}$ and $\frac{1}{47}$.

6. If a, b, c, d are in harmonic progression, show that

$$ab : cd :: a-b : c-d.$$

7. Show that the geometric mean of two numbers is also the geometric mean of their arithmetic and harmonic means.

CHAPTER XV

QUADRATIC EQUATIONS

338. **Quadratic Equations** (Art. 238) are distinguished as **Pure** (called also **Incomplete**) and **Affected** (called also **Complete**).

339. **A Pure Quadratic Equation** is an equation which contains no power of the unknown quantity but the second.

Thus, $ax^2 + b = cd$ and $3x^2 = 108$ are pure quadratic equations, or pure quadratics.

340. **An Affected Quadratic Equation** is an equation which contains both the first and the second powers of the unknown quantity.

Thus, $x^2 - 4x = 12$, $2x^2 + 7x - 18 = 0$, $ax^2 + bx = c$, are affected quadratic equations, or affected quadratics.

341. **A Root of an Equation** is a quantity which, substituted for the unknown quantity, satisfies the equation.

PURE QUADRATICS

342. **Prob.** *To solve a pure quadratic equation.*

Rule. *By clearing of fractions, transposing, uniting terms, and dividing by the coefficient of the square of the unknown quantity, reduce the equation to the form $x^2 = m$; then extract the square root of both members, giving the double sign to the second member of the result.*

Dem. All of the operations of reducing to the form $x^2 = m$, and the extraction of the square root as well, affect both members alike, and, consequently (Art. 243), do not destroy the equality of the members. As these operations leave in the first member simply the unknown quantity, the equation is solved.

That the result should have the double sign is evident from the fact that a quantity has two square roots numerically equal, but with opposite signs.

343. Cor. *The roots of a pure quadratic equation are both rational or both surd, both real or both imaginary.*

EXAMPLES LXXXII

Solve the following:

1. $11x^2 - 44 = 5x^2 + 10.$

2. $5x^2 - 9 = 2x^2 + 66.$

3. $(x+2)^2 = 4x + 5.$

4. $\dfrac{5}{4-x} = \dfrac{8}{3} - \dfrac{5}{4+x}.$

5. $\dfrac{8}{1-2x} + \dfrac{8}{1+2x} = 25.$

6. $2(x+3)(x-3) = (x+1)^2 - 2x.$

7. $x^2 - ax + b = ax(x-1).$

8. $x\sqrt{x^2+6} = x^2 + 1.$

9. $\dfrac{a}{x} + \dfrac{\sqrt{a^2 - x^2}}{x} = \dfrac{x}{b}.$

10. $x + \sqrt{a^2 + x^2} = \dfrac{2a^2}{\sqrt{a^2 + x^2}}.$

11. $\dfrac{2}{x+\sqrt{2-x^2}} + \dfrac{2}{x-\sqrt{2-x^2}} = x.$

12. $\dfrac{ax + 1 + \sqrt{a^2x^2 - 1}}{ax + 1 - \sqrt{a^2x^2 - 1}} = \dfrac{bx}{2}.$

PROBLEMS LEADING TO PURE QUADRATIC EQUATIONS

EXAMPLES LXXXIII

1. Find two numbers in the ratio of 2 to 5, the sum of whose squares is 261.

2. Find three numbers which shall be to one another as m, n, and p, and the sum of whose squares shall be s.

3. Divide 21 into two such parts that the square of the less shall be to that of the greater as 4 to 25.

4. The sides of two square rooms are in the ratio of 2 to 3, and the larger room requires 20 square yards more of carpet than the smaller. Find a side of each.

5. Two square plats of ground contain 272 square rods, and a side of the larger is as much greater than 10 rods as a side of the other is less than 10 rods. Find a side of each.

6. A rectangular field whose length is $1\frac{1}{9}$ times its breadth contains 9 acres. Find the length, in rods, of each side.

7. An army was formed with 5 more men in file than in rank; but when the form was changed so that there were 845 more men in rank than before, there were but 5 ranks. Find the number of men in the army.

8. A boat's crew can row in still water at the rate of 9 miles an hour. If it take the crew $2\frac{1}{4}$ hours to row 9 miles up a river and back to the starting point, what is the rate of the current?

9. The distances through which a body falls being as the squares of the times, and the distance fallen during the first second being $16\frac{1}{12}$ feet, in what time will a body fall 500 feet? In what time will it fall a mile?

10. The mass of the earth is 332,000 times that of the earth, and the distance between the two bodies is 93,000,000 miles. How far from the earth, between the earth and the sun, is the point of equal attraction, the law of attraction being that it varies directly as the mass and inversely as the square of the distance?

SUG. In this and some of the following problems, to avoid an affected quadratic equation, take the square root before performing indicated operations.

11. The intensities of two lights are as $7:17$, and their distance apart 132 feet. Where in the line of the lights are the points of equal illumination, assuming that the amounts of illumination are to each other directly as the intensities and inversely as the squares of the distances?

12. The loudness of one church bell is three times that of another. If the amount of sound varies directly as the loudness and inversely as the square of the distance, where on the line of the two will the bells be equally well heard, the distance between them being a?

13. A girl worked two square pieces of worsted work of the same kind, the edges of one being an inch longer than those of the other; one took $12\frac{1}{2}$ skeins and the other 18 skeins. How long were the edges of the smaller piece?

14. From two towns m miles apart, two persons, A and B, started at the same time and traveled toward each other. When they met, A, the faster traveler, had gone n miles, the time on the road being equal to the difference of their rates. Find their rates.

15. A and B are two stations 300 miles apart. Two trains start simultaneously from A and B, each to the opposite station. The train from A reaches B 9 hours, and the train from B reaches A 4 hours, after they meet. Find the rate of each train.

16. Two travelers, A and B, started at the same time from two different places, C and D respectively, and traveled toward each other. When they met, it appeared that A had gone 30 miles more than B; also that A could reach D in 4 days, and B could reach C in 9 days. Find the distance from C to D.

17. Two bicyclists start at the intersection of two roads at right angles to each other and ride, one on each road, at rates of 12 and 16 miles an hour respectively. In how many minutes will they be 8 miles apart?

AFFECTED QUADRATICS

344. Prob. *To solve an affected quadratic equation.*

RULE. *Reduce to the form $x^2 + px = q$ (in which p and q may be positive or negative, integral or fractional); then the roots are half of the coefficient of the second term taken with the opposite sign, $\pm$ the square root of the sum of the square of this half coefficient and the absolute term.*

DEM. All of the operations of reducing to the form

$$x^2 + px = q,$$

viz., clearing of fractions, transposing, uniting terms, and dividing by the coefficient of the second power of the unknown quantity, affect both members alike, and, consequently (Art. 243), do not destroy the equality of the members.

Now if $\frac{p^2}{4}$ be added to both members of the equation, giving

$$x^2 + px + \frac{p^2}{4} = \frac{p^2}{4} + q,$$

the equality will not be destroyed and the first member will be a perfect square, since the middle term is twice the product of the square roots of the other two (Arts. 61 and 62). This operation is called *completing the square.* Extracting the square root of both members, which does not destroy the equality, we have

$$x+\frac{p}{2}=\pm\sqrt{\frac{p^2}{4}+q},$$

or

$$x=-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}+q},$$

which corresponds with the statement in the rule.

345. Cor. 1. *If the first term of the roots is numerically greater than the radical term, both roots have the sign of the first term; if numerically less, one root is plus and one minus. If q is negative and numerically greater than $\frac{p^2}{4}$, both roots are imaginary. If q is negative and numerically equal to $\frac{p^2}{4}$, the two roots are equal.*

346. Cor. 2. *The sum of the two roots is $-p$ and the product is $-q$.*

EXAMPLES LXXXIV

Solve the following:

1. $3x^2-29=18x-8.$

Solution. Transposing, uniting terms, and dividing by the coefficient of x^2, this becomes

$$x^2-6x=7.$$

The student should not complete the square — that has been done once for all in the demonstration of the rule — but should write the roots at once from the formula obtained by solving the general case. The roots being *half of the coefficient of the second term taken with the opposite sign, $\pm$ the square root of the sum of the square of this half coefficient and the absolute term*, we have from

$$x^2-6x=7,$$

$$x=3\pm 4=7 \text{ or } -1.$$

The full form is

$$x=3\pm\sqrt{3^2+7}=3\pm 4=7 \text{ or } -1;$$

but when the numbers are small, the operations are readily performed mentally.

2. $15 - \dfrac{5x^2}{2} = 12\frac{1}{2}x.$

SOLUTION. Reduced to the required form, this becomes

$$x^2 + 5x = 6,$$

whence $$x = -\tfrac{5}{2} \pm \tfrac{7}{2} = 1 \text{ or } -6.$$

3. $x^2 - 4x = 5.$ **4.** $x^2 + 6x = -5.$ **5.** $x^2 - 10x = 11.$

6. $x^2 + 8x = 9.$ **7.** $x^2 - 12x = -11.$ **8.** $x^2 - 2x = 48.$

9. $x^2 - 6x = 16.$ **10.** $3x^2 + 36 = 24x.$ **11.** $x^2 - 4x = 60.$

12. $x^2 - 7x = 8.$ **13.** $x^2 - 30x = 64.$ **14.** $x^2 - 12x = 28.$

15. $x^2 - 12x = 45.$ **16.** $x^2 - 8x = 33.$ **17.** $6x^2 - 21x = 12.$

18. $\dfrac{x}{x-1} = \dfrac{3}{2} + \dfrac{x-1}{2}.$ **19.** $\dfrac{a^2x^2}{b^2} - \dfrac{2ax}{c} + \dfrac{b^2}{c^2} = 0.$

20. $\dfrac{x+2}{5} = \dfrac{8}{3x+4}.$ **21.** $x^2 + \dfrac{a^2 - b^2}{ab}x = 1.$

22. $cx^2 + \dfrac{a^2 - b^2}{c} = 2ax.$ **23.** $\dfrac{2x(b-x)}{3b - 2x} = \dfrac{b}{4}.$

PROBLEMS LEADING TO AFFECTED QUADRATIC EQUATIONS

347. When a problem gives rise to a quadratic equation, the student should note whether both values obtained by the solution of the equation are admissible and capable of interpretation. A problem often contains restrictions, expressed or implied, which cannot be incorporated in the equation, and hence the equation may contain a root that will not conform to this restriction. For example, in a problem involving as the unknown quantity the digits of a number, sheep, cattle, men, etc., any fractional root must be rejected. In some problems negative roots must be rejected. Imaginary roots indicate incompatibility of conditions in the problem (Art. 227).

EXAMPLES LXXXV

1. In a number consisting of two digits the tens' digit is 1 less than the square of the units' digit, and when 45 is subtracted from the number, the digits change places. Find the number.

SOLUTION. Let $x =$ the units' digit;

then $(x^2 - 1) =$ the tens' digit,

and we have from the conditions,

$$10(x^2 - 1) + x - 45 = 10x + x^2 - 1,$$

or $$x^2 - x = 6,$$

whence $$x = \tfrac{1}{2} \pm \tfrac{5}{2} = 3 \text{ or } -2.$$

The root -2 must be rejected, as the digits of a number must be positive. Hence the units' digit is 3, the tens' digit is $3^2 - 1 = 8$, and the number is 83.

2. Two boys are coaching for an examination in Greek. One of them reads a page the first day, and each succeeding day reads one more page than the day before. The other, beginning 5 days later, reads 12 pages a day. In how many days will they have read the same number of pages?

SOLUTION. Let $x =$ the first boy's time;

then $x - 5 =$ the second boy's time,

and $(x - 5)12 =$ the number of pages read by the second boy.

The number of pages read by the first boy is the sum of an arithmetical progression in which the first term is 1, the number of terms x, and the last term x; *i.e.* (Art. 319),

$$\left(\frac{a + l}{2}\right)n = \left(\frac{1 + x}{2}\right)x.$$

Hence $$\left(\frac{1 + x}{2}\right)x = (x - 5)\,12,$$

or $$x^2 - 23x = -120,$$

and $$x = \frac{23}{2} \pm \frac{7}{2} = 15 \text{ or } 8.$$

Both roots of the equation are admissible. The second boy overtakes the first in 8 days and is overtaken by him in 15 days (from the beginning), since the first is reading at an increasing rate.

3. Divide 48 into two such parts that their product may be 432.

4. Divide 24 into two such parts that their product may be 35 times their difference.

5. For a journey of 108 miles, 6 hours less would have sufficed, had the traveler gone 3 miles an hour faster. At what rate did he travel?

6. The length of a rectangle is 10 feet more than the breadth, and the area is 600 square feet. Find the length and breadth of the rectangle.

7. If a bar of iron weighing 60 pounds be drawn out 3 feet longer, it will weigh 1 pound less per linear foot. Find its length.

8. A man bought two farms for $ 2800 each. The larger contained 10 acres more than the smaller, but he paid $ 5 more per acre for the smaller than for the larger. How many acres did each contain?

9. A man paid $ 300 for a drove of sheep. By selling all but 10 of them at a profit of $ 2.50 each, he received the amount he paid for all the sheep. How many sheep did he buy?

10. It takes a boat's crew 4 hours and 12 minutes to row 12 miles down a river and back. If the rate of the current is 3 miles an hour, at what rate can the crew row in still water?

11. Two steamers ply between the same two ports a distance of 420 miles. One goes $\frac{1}{2}$ mile per hour faster than the other, and is 2 hours less on the voyage. At what rates do they go?

12. The plate of a mirror, 18 inches by 12, is to be surrounded by a plain frame whose surface shall be equal to that of the glass. Find the width of the frame.

13. A man bought some sheep for $ 360, and his neighbor bought 6 more for the same amount, paying $ 5 less for each. How many did the first man buy, and what was the price of each?

14. A man bought shares in a company for $ 375. A later investor, after the shares had declined $ 6.25 each, bought for the same amount five more than did the first man. How many shares did the first man buy?

15. A battalion of soldiers, when formed into a solid square, presents 16 men fewer in the front than when formed into a hollow square four deep. Required the number of men.

16. Two vessels, one of which sails faster than the other by 2 miles an hour, start together for different ports. The faster vessel completes its voyage of 1152 miles 1 day later than the other completes its voyage of 720 miles. What is the rate of the faster vessel?

17. A regiment received orders to send 216 men on garrison duty, each company sending the same number of men; but before the detachments marched, three entire companies were sent on other service, and it was then found that each remaining company would have to send 12 men additional to furnish the required 216. How many companies were in the regiment, and how many men did each remaining company send on garrison duty?

18. Two trains simultaneously leave A and B, which are 81 miles apart, and pass each other in 1 hour. The train from A reaches B 27 minutes earlier than the one from B reaches A. Find the time of each train.

19. A boat goes along a straight reach of a canal at 6 miles an hour. A person living 4 miles from the canal sets out, three quarters of an hour before it is due at it nearest point to his residence, to catch the boat. If he goes 4 miles an hour, find how far below the nearest point of the canal is the point toward which he must direct his course, in order that he may reach it just with the boat.

20. A starts at 10 A.M. to walk from P to Q, and B starts at 10:24 A.M. to walk from Q to P. They meet 6 miles from Q. B stops 1 hour at P, and A stops 2 hours and 54 minutes at Q, and returning they meet midway between P and Q at 6:54 P.M. Find the distance from P to Q.

348. Theorem. *If all the terms of an equation that is integral with reference to the unknown quantity be transposed to the first member, and the resulting polynomial be resolved into factors containing the unknown quantity, the values obtained by placing these factors in turn equal to* 0 *are the roots of the original equation.*

DEM. By hypothesis the second member of the equation is 0. Now since any finite quantity multiplied by 0 is 0, the placing of any one of the factors equal to 0 will render the first member 0, and the equation will be satisfied. Hence the values which render the factor 0 will be roots of the original equation.

If the factors are of the first degree in the form $x-a$, $x-b$, $x-c$, $\cdots$ $x-l$, the equation is

$$(x-a)(x-b)(x-c)\cdots(x-l)=0,$$

and from $x-a=0$, $x-b=0$, $x-c=0$, etc., we have by transposition $x=a$, $x=b$, $x=c$, etc.

349. Cor. Conversely, *to form an equation having given roots, it is but necessary to subtract the roots from x and place the product of the remainders equal to* 0.

Thus, to form an equation whose roots are 1, 3, and -5, *i.e.*, an equation which is satisfied when $x=1$, $x=3$, and $x=-5$, we have by transposition $x-1=0$, $x-3=0$, and $x+5=0$, and by multiplying these equations together (using the method of Art. 59), $x^3+x^2-17x+15=0$.

If but two roots be given, the required quadratic may be written at once by Art. 346. Thus, if the roots are 2 and 3,

$$p=-(2+3)=-5; q=-(2\times 3)=-6,$$

and the equation is $x^2-5x=-6$.

EXAMPLES LXXXVI

Solve the following by Art. 348.

1. $x^2-5x=-6$.

SOLUTION. Transposing, $\quad x^2-5x+6=0.$

Factoring, $\quad (x-2)(x-3)=0.$

This is satisfied when $x-2=0$, and when $x-3=0$, giving $x=2$ and $x=3$.

2. $x^2-5x=14$.

SOLUTION. $x^2-5x-14=(x+2)(x-7)=0$; $\therefore x=-2$ or 7.

3. $x^2-6x+5=0$. **4.** $x^2-5x=24$. **5.** $x^2-6x+9=0$.

6. $x^2=49$. **7.** $x^2-7x=0$. **8.** $x^2-12x=-36$.

Form the equations whose roots are the following, either writing the products by Art. 60, or the equations at once by Art. 346:

9. 4 and -2. **10.** 7 and 3. **11.** -8 and 5.

12. -6 and -9. **13.** $\sqrt{3}$ and $-\sqrt{3}$. **14.** $2+\sqrt{5}$ and $2-\sqrt{5}$.

15. By a misprint in an examination paper the absolute term of a quadratic equation in the form $x^2+px=q$ is made $+$ instead of $-$, thus making the roots 12 and -2. What are the roots of the equation in the copy?

16. Two boys attempt to solve a quadratic equation. After reducing it to the form $x^2 + px = q$, one of them has a mistake only in the absolute term, and finds the roots to be 1 and 7; the other has a mistake only in the coefficient of x, and finds the roots to be -1 and -12. Find the roots of the correct equation.

17. Find the relation that must exist between p and q in order that one root of $x^2 + px + q = 0$ may be double of the other.

18. What condition will make one root of $x^2 + px = q$ the reciprocal of the other?

350. Art. 348 has not been introduced in connection with affected quadratic equations for the purpose of replacing the method of solution given in Art. 344, but for the purpose of showing how roots are sometimes gained and sometimes lost in transforming equations, and for providing against such gain or loss.

351. Theorem. *Multiplying an equation by a factor containing the unknown quantity introduces into the resulting equation the root or roots obtained by placing the factor equal to* 0; *and*, conversely, *removing from an equation a factor containing the unknown quantity removes from the equation the root or roots obtained by placing the factor equal to* 0.

Dem. 1st. The terms all being transposed to the first member, the second member is 0; and if the introduced factor be made equal to 0, the product will be 0, and the new equation will be satisfied. Hence the values which render the introduced factor 0 are roots of the new equation.

2d. If an equation be divided by a factor containing the unknown quantity, it loses those values of the unknown quantity which, by rendering that factor 0, satisfy the original equation.

352. Sch. In clearing of fractions an equation having the unknown quantity in the denominator of some of the terms, to avoid the introduction of roots that do not belong to the original equation, sometimes called **Extraneous Roots**, care should be taken to multiply by the *lowest* common denominator; and when the

degree of an equation is reduced by dividing out a factor, the roots obtained by placing this factor equal to 0 must be retained as roots of the original equation.

ILLUSTRATIVE EXAMPLES

1. Let it be required to find the roots of

$$\frac{5}{x^2-1}-\frac{2}{x+1}=\frac{1}{x-1}.$$

When we clear of fractions by multiplying by the l. c. m. of the denominators and reduce, we have

$$3(x-2)=0,$$

and $x-2$ is the only factor that can be placed equal to 0, giving $x=2$ as the only root.

When we clear of fractions by multiplying by the product of all the denominators and reduce, we have

$$3(x^2-1)(x-2)=0,$$

which is satisfied not only for $x-2=0$, but also for $x^2-1=0$, giving $x=2$, 1, or -1. The roots 1 and -1 do not belong to the original equation, but entered by the introduction of the extra factor x^2-1 in clearing of fractions.

2. Let it be required to find the roots of

$$x^2-6x-7=3\sqrt{x^2-6x-7}.$$

Dividing by $\sqrt{x^2-6x-7}$, we have

$$\sqrt{x^2-6x-7}=3.$$

Squaring both members and solving, we have $x=8$ or -2. These numbers satisfy the original equation, but so also do 7 and -1. How, then, did these two roots escape?

The original equation may be put in the form

$$\sqrt{x^2-6x-7}(\sqrt{x^2-6x-7}-3)=0,$$

and this equation is satisfied by placing either factor equal to 0, and each of these equations gives two roots. Two of them, 7 and -1, were lost in dividing out the factor $\sqrt{x^2-6x-7}$.

If we choose, we may free the equation of radicals, giving an equation of the 4th degree, viz.,

$$x^4-12x^3+13x^2+138x+112=0,$$

and factor by Art. 101, giving

$$(x-8)(x+2)(x-7)(x+1)=0,$$

the four roots of which are seen to be 8, -2, 7, and -1.

353. In verifying a radical equation the double sign of a radical term is not always admissible. In the last illustrative example, if 8 be substituted for x in the equation

$$x^2 - 6x - 7 = 3\sqrt{x^2 - 6x - 7},$$

the first member becomes 9 and the second member becomes

$$3\sqrt{9} = 3(\pm 3) = \pm 9;$$

and, as $+9$ cannot equal -9, only the upper sign is admissible. The same is true when -2 is substituted; but, when 7 or -1 is substituted, both signs are admissible, as in each case each member vanishes. In such an expression as $x - 2 = \sqrt{x + 11}$ there are really two equations, viz.,

$$x - 2 = +\sqrt{x + 11} \text{ and } x - 2 = -\sqrt{x + 11}.$$

Each gives, when squared and reduced, $x^2 + x = 12$; whence, $x = 3$ or -4. One of these values satisfies the first of the given equations, the other satisfies the second, and both satisfy the equation that results from squaring.

354. When, to free it of radicals or for other purpose, an equation is squared, there is no way of indicating in the resulting equation itself that there is any restriction in the matter of signs. For example, if we square both members of $x = 3$, giving $x^2 = 9$, and then solve this equation, we obtain $x = 3$ or -3. If, however, we transpose before squaring, we have $x - 3 = 0$, $x^2 - 6x + 9 = 0$; whence, $x^2 - 6x = -9$ and $x = 3 \pm 0 = 3$. Hence, additional roots may or may not be introduced into an equation by squaring it. It follows that *when an equation is rendered rational by squaring it, some of the roots may have to be rejected.* A trial of the roots in the original equation will determine what ones are to be retained.

EXAMPLES LXXXVII

Solve the following, rejecting extraneous roots:

1. $\sqrt{5x + 1} = \sqrt{x + 1} + 2.$
2. $\sqrt{7x + 14} = \sqrt{2x + 6} + \sqrt{x + 4}.$

3. $2\sqrt{x^2 - 9x + 18} - \sqrt{x^2 - 4x - 12} = x - 6.$

4. $\sqrt{2x^2 + 7x - 9} - \sqrt{x^2 - 5x + 4} = \sqrt{x^2 - 1}.$

5. $\sqrt{2x^2 + 10x + 8} - \sqrt{x^2 + 6x + 5} = \sqrt{x + 1}.$

6. $2\sqrt{x} + \sqrt{4x + \sqrt{7x + 2}} = 1.$

7. $\sqrt{ax + b^2} + \sqrt{bx + a^2} = a - b.$

8. $\sqrt{(x-1)(x-2)} + \sqrt{(x-3)(x-4)} = \sqrt{2}.$

9. $2\sqrt{x} + \dfrac{2}{\sqrt{x}} = 5.$

10. $\dfrac{x - \sqrt{x+1}}{x + \sqrt{x+1}} = \dfrac{5}{11}.$

11. $\dfrac{5(3x - 1)}{1 + 5\sqrt{x}} + \dfrac{2}{\sqrt{x}} = 3\sqrt{x}.$

12. $\dfrac{1}{\sqrt{2 + x} - \sqrt{2}} + \dfrac{1}{\sqrt{2 - x} + \sqrt{2}} = \dfrac{\sqrt{2}}{x}.$

13. $\dfrac{x + \sqrt{x^2 - 9}}{x - \sqrt{x^2 - 9}} = (x - 2)^2.$

14. $\dfrac{\sqrt{1 + x}}{1 + \sqrt{1 + x}} = \dfrac{\sqrt{1 - x}}{1 - \sqrt{1 - x}}.$

15. $\sqrt{4 + \sqrt{2x^3 + x^2}} = \dfrac{x + 4}{2}.$

16. $\dfrac{x + \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}} + \dfrac{x - \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}} = 98.$

CHAPTER XVI

SOME HIGHER EQUATIONS

PURE EQUATIONS

355. A Pure Equation is an equation in which the unknown quantity is affected with but one exponent.

356. Prob. *To solve any pure equation.*

RULE. *Reduce to the form in which the affected unknown quantity with coefficient unity shall constitute one member of the equation; then perform upon both members the operations necessary to make the exponent of the unknown quantity unity.*

DEM. 1st. When the form after reduction is $x^m = a$.

By extracting the mth root of both members we have $x = \sqrt[m]{a}$.

2d. When the reduced form is $x^{\frac{m}{n}} = a$.

By extracting the mth root of both members and raising to the nth power both members of the resulting equation, we have

$$x = (\sqrt[m]{a})^n.$$

3d. When the reduced form is $x^{-m} = a$.

By Art. 147, 2d, this is the same as $\dfrac{1}{x^m} = a$; whence $x^m = \dfrac{1}{a}$, and $x = \sqrt[m]{\dfrac{1}{a}}$.

357. NOTE. When both evolution and involution are necessary, as in freeing of a fractional exponent, it is expedient to use evolution first, thus avoiding large numbers.

It will be shown in a subsequent part of the work that an equation has as many roots as indicated by its degree. In the following examples only those roots that are obtained by the above process are required.

EXAMPLES LXXXVIII

Solve the following:

1. $x^4 = 256.$ **2.** $x^5 = 243.$ **3.** $x^{\frac{3}{2}} = 216.$

4. $x^{\frac{2}{3}} = 81.$ **5.** $x^{\frac{4}{3}} = 625.$ **6.** $x^{\frac{5}{3}} = 243.$

7. $x^{-3} = \frac{27}{343}.$ **8.** $x^{-\frac{5}{2}} = \frac{32}{243}.$ **9.** $\sqrt[3]{x^4} = 256.$

358. *The rule of Art. 356 applies when, instead of a simple quantity, as x, there is a group of terms affected as a whole with an exponent.*

EXAMPLES LXXXIX

Solve the following:

1. $x^6 - 6x^4 + 12x^2 - 8 = 0.$

Solution. This is not a pure equation with reference to x, but is a pure equation with reference to $x^2 - 2$. Regarding $x^2 - 2$ as the unknown quantity, we have

$$x^6 - 6x^4 + 12x^2 - 8 = (x^2 - 2)^3 = 0,$$

whence $$x^2 - 2 = 0, \text{ and } x = \pm\sqrt{2}.$$

2. $(x^3 - 6)^5 = 32.$ **3.** $(x^2 - 6x - 2)^3 = 125.$

4. $(x^2 + 2)^{\frac{4}{3}} = 81.$ **5.** $(x^{-2} + \frac{1}{4})^{-3} = 8.$

6. $x^3 + 9x^2 + 27x + 27 = 0.$ **7.** $x^6 - 12x^4 + 48x^2 = 64.$

8. $x^4 - 10x^3 + 35x^2 - 50x + 25 = 0.$

9. $64x^6 + 96x^5 - 96x^4 - 136x^3 + 72x^2 + 54x - 27 = 0.$

10. $(x + 3)^6 = (2x^2 + 10x + 4)^3.$

EQUATIONS IN THE QUADRATIC FORM

359. An equation is in the **Quadratic Form** when the unknown quantity has but two exponents, one of which is twice the other.

360. Prob. *To solve an equation in or reducible to the quadratic form.*

Solution. The reduced general form being $x^{2m} + px^m = q$, we may write it $(x^m)^2 + px^m = q$, which is an affected quadratic equa-

tion with reference to x^m. Regarding x^m as the unknown quantity, we have, by Art. 344,

$$x^m = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q},$$

whence $$x = \sqrt[m]{-\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}}.$$

EXAMPLES XC

Solve the following:

1. $x^6 + 7x^3 = 8.$

SOLUTION. Writing the equation in the form

$$(x^3)^2 + 7x^3 = 8,$$

it is seen to be an affected quadratic, not with reference to x, but with reference to x^3. Regarding x^3 as the unknown quantity and solving by Art. 344, we have

$$x^3 = -\tfrac{7}{2} \pm \tfrac{9}{2} = 1 \text{ or } -8,$$

whence $$x = 1 \text{ or } -2.$$

2. $x^4 - 5x^2 = -4.$ **3.** $x^4 - 6x^2 = -5.$ **4.** $x^4 - 8x^2 = 9.$

5. $x^4 - 25x^2 = -144.$ **6.** $x^6 + 6x^3 = 16.$ **7.** $x^3 - x^{\frac{3}{2}} = 56.$

8. $x^{\frac{1}{4}} + 5x^{\frac{1}{2}} = 22.$ **9.** $x^{\frac{6}{5}} + x^{\frac{3}{5}} = 756.$ **10.** $x^{\frac{1}{3}} + \dfrac{5}{2x^{\frac{1}{3}}} = \dfrac{13}{4}.$

11. $5x^{-4} - 11x^{-2} = -6.$ **12.** $x^{\frac{5}{6}} + x^{\frac{5}{3}} = 1056.$

13. $\dfrac{1}{x^{2n}} + \dfrac{3}{x^n} = 18.$ **14.** $3x^n\sqrt[3]{x^n} + \dfrac{2x^n}{\sqrt[3]{x^n}} = 16.$

15. $x^{\frac{1}{4}} - x^{-\frac{1}{4}} = \frac{3}{2}.$ **16.** $x^{-3} + x^3 = \frac{65}{8}.$

361. *The solution of Art. 360 applies when, instead of a simple quantity, as x, there is a group of terms having as a whole only two exponents, one of which is twice the other.*

EXAMPLES XCI

Solve the following:

1. $x^2 - 4x - 6\sqrt{x^2 - 4x - 5} = -3.$

SOLUTION. Subtracting 5 from both members, we have

$$x^2 - 4x - 5 - 6\sqrt{x^2 - 4x - 5} = -8.$$

Here we have a group of terms, minus 6 times the square root of the group, or, what is the same thing, a group of terms with the exponent 1, minus 6 times the same group with the exponent $\frac{1}{2}$. Hence, regarding $\sqrt{x^2 - 4x - 5}$ as the unknown quantity and solving as an affected quadratic (Art. 344), we have

$$\sqrt{x^2 - 4x - 5} = 3 \pm 1 = 4 \text{ or } 2.$$

Squaring, etc.,
$$x^2 - 4x - 5 = 16 \text{ or } 4,$$
$$x^2 - 4x = 21 \text{ or } 9,$$
$$x = 2 \pm 5 \text{ or } 2 \pm \sqrt{13};$$
$$\therefore\ x = 7 \text{ or } -3 \text{ or } 2 \pm \sqrt{13}.$$

The last two roots do not satisfy the original equation.

2. $2x^2 - 5x - 2\sqrt{2x^2 - 5x} = 15.$

3. $x^2 - 2x + 6\sqrt{x^2 - 2x + 5} = 11.$

4. $3(x+7)^{\frac{1}{2}} + 3(x+7)^{-\frac{1}{2}} = 10.$

5. $x - 3\sqrt{x + 16} = -6.$

6. $x^2 - \sqrt{x^2 - 9} = 21.$

7. $2x^2 - 5x + 5\sqrt{2x^2 - 5x + 6} = 33.$

8. $x^2 + 3x - \sqrt{2x^2 + 6x + 1} = 1.$

9. $\sqrt{x + 16} + \sqrt[4]{x + 16} = 6.$

10. $x^2 + \sqrt{x^2 - 7x + 8} = 7x + 4.$

11. $2x^2 - 7x + 2\sqrt{2x^2 - 7x + 6} = -6.$

12. $2x^2 + 6\sqrt{2x^2 - 3x + 2} = 3x + 14.$

13. $2x^2 - 4x + 3\sqrt{x^2 - 2x + 6} = 15.$

14. $\left(x - \frac{1}{x}\right)^2 + 7\left(x - \frac{1}{x}\right) = 12\frac{3}{4}.$

15. $x^2 + \frac{1}{x^2} + x + \frac{1}{x} = 4.$

16. $\frac{x^2 - 3x - 3}{7} - \frac{1}{x^2 - 3x - 3} = \frac{6}{7}.$

17. $x^4\left(1 + \frac{1}{3x}\right)^2 - (3x^2 + x) = 70.$

18. $\sqrt{x - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}} = x.$

19. $1 + \sqrt{1 - \frac{a}{x}} = \sqrt{1 + \frac{x}{a}}.$

EQUATIONS WITH INTEGRAL ROOTS

362. An equation containing one unknown quantity is said to be in the **Normal** or **Typical Form** when the exponents are all positive integers, the coefficient of the highest power is 1, and the other coefficients are integers.

This form is called *normal* or *typical* because, as shown in a subsequent part of the work, every equation having rational coefficients can be reduced to it.

363. Prob. *To find the integral roots of an equation in the typical form.*

Solution. If the polynomial resulting from transposing all the terms to the first member has integral factors of the first degree, they may be found by the process of Art. 101. Now, by Art. 348, the roots obtained by placing these factors equal to 0 are roots of the original equation.

If all the roots but two are found by this process, the remaining two, even if surd or imaginary, may be found by placing the remaining factor, which is quadratic, equal to 0.

364. Cor. *Some or all of the roots may be equal.*

Note. In solving the following examples carefully observe the instructions of Art. 102.

EXAMPLES XCII

Find the roots of the following:

1. $x^5 - x^4 - 13x^3 + 13x^2 + 36x - 36 = 0.$

Operation. Proceeding as in Art. 101, using the smaller divisors of 36 (only factors of the absolute term need be tried in any case), we have the following:

$$\begin{array}{cccccc|l} x^5 & -x^4 & -13x^3 & +13x^2 & +36x & -36 & \underline{\ 1\ } \\ & 0 & -13 & 0 & 36 & 0 & \underline{\ 2\ } \\ & 2 & -9 & -18 & 0 & & \underline{\ 3\ } \\ & 5 & 6 & 0 & & & \underline{-2} \\ & 3 & 0 & & & & \underline{-3} \\ & 0 & & & & & \end{array}$$

Therefore the roots are 1, 2, 3, -2, -3. This is because the factors are $x-1$, $x-2$, $x-3$, $x+2$, and $x+3$, and when these are placed equal to 0 (Art. 348), we have by transposition

$$x=1,\ x=2,\ x=3,\ x=-2,\ x=-3.$$

2. $x^3-9x^2+26x-24=0.$

3. $x^3+5x^2-9x-45=0.$

4. $x^3-8x^2+13x-6=0.$

5. $x^3+2x^2-23x-60=0.$

6. $x^3-x^2-8x+12=0.$

7. $x^3-5x^2-8x+48=0.$

8. $x^3+8x^2+20x+16=0.$

9. $x^3-13x^2+47x-35=0.$

10. $x^4-3x^3-14x^2+48x-32=0.$

11. $x^4-11x^2+18x-8=0.$

SUG. Supply the missing term, with coefficient 0.

12. $x^4-45x^2-40x+84=0.$

13. $x^4+13x^3+33x^2+31x+10=0.$

QUERY. Why is it unnecessary to try any positive numbers?

14. $x^6-3x^5+6x^3-3x^2-3x+2=0.$

OPERATION

$$\begin{array}{ccccccc|c}
x^6 & -3x^5 & +0x^4 & +6x^3 & -3x^2 & -3x & +2 & 1 \\
 & -2 & -2 & 4 & 1 & -2 & 0 & 1 \\
 & -1 & -3 & 1 & 2 & 0 & & 1 \\
 & 0 & -3 & -2 & 0 & & & -1 \\
 & -1 & -2 & 0 & & & & -1 \\
 & -2 & 0 & & & & & 2 \\
 & 0 & & & & & &
\end{array}$$

Hence the roots are 1, 1, 1, -1, -1, 2.

15. $x^5-2x^4-15x^3+8x^2+68x+48=0.$

16. $x^5-13x^4+67x^3-171x^2+216x-108=0.$

17. $x^6+3x^5-6x^4-6x^3+9x^2+3x-4=0.$

18. $x^7+5x^6+6x^5-6x^4-15x^3-3x^2+8x+4=0.$

19. $x^5-8x^4+21x^3-16x^2-10x+12=0.$

Operation

$$\begin{array}{cccccc|l}
x^5 - 8\,x^4 & + 21\,x^3 & - 16\,x^2 & - 10\,x & + 12 & & \underline{1} \\
-7 & 14 & -2 & -12 & 0 & & \underline{2} \\
-5 & 4 & 6 & 0 & & & \underline{3} \\
-2 & -2 & 0 & & & &
\end{array}$$

Further trial by this process fails to give the other two roots. (There must be five roots (Art. 357), since the equation is of the fifth degree.) However, the factors of the polynomial are $x-1$, $x-2$, $x-3$, and $x^2-2\,x-2$, the last row of numbers, $1, -2, -2$, furnishing the coefficients for the last one (Art. 81). Placing the quadratic factor equal to 0 (Art. 348), we have

$$x^2 - 2\,x - 2 = 0,$$

or

$$x^2 - 2\,x = 2,$$

whence

$$x = 1 \pm \sqrt{3}.$$

Hence the roots are $1, 2, 3, 1 \pm \sqrt{3}$.

20. $x^3 - 6\,x^2 + 10\,x - 8 = 0$. **21.** $x^3 - 3\,x^2 + x + 2 = 0$.

22. $x^4 - 6\,x^3 + 24\,x - 16 = 0$. **23.** $x^4 - 4\,x^3 - 8\,x + 32 = 0$.

24. $x^4 - 9\,x^3 + 17\,x^2 + 27\,x - 60 = 0$.

25. $x^5 - 3\,x^4 - 9\,x^3 + 21\,x^2 - 10\,x + 24 = 0$.

26. $x^5 - 4\,x^4 - 16\,x^3 + 112\,x^2 - 208\,x + 128 = 0$.

27. $x^6 - 7\,x^5 + 11\,x^4 - 7\,x^3 + 14\,x^2 - 28\,x + 40 = 0$.

365. The method of Art. 363 applies also when the coefficient of the first term is not 1; but in that case, if there is no monomial factor common to all the terms, the rational roots are not all integral divisors of the absolute term. We should find first any integral roots the equation may contain, and find the remaining root or roots by placing the remaining factor equal to 0.

EXAMPLES XCIII

Find the roots of the following:

1. $3\,x^4 - 8\,x^3 - 11\,x^2 + 28\,x - 12 = 0$.

Operation

$$\begin{array}{ccccc|l}
3\,x^4 - 8\,x^3 & - 11\,x^2 & + 28\,x & - 12 & & \underline{1} \\
-5 & -16 & 12 & 0 & & \underline{3} \\
4 & -4 & 0 & & & \underline{-2} \\
-2 & 0 & & & &
\end{array}$$

The last factor of the polynomial is seen to be $3x-2$; and by Art. 348,

$$3x-2=0,$$

whence $$x=\tfrac{2}{3}.$$

Hence the roots are 1, 3, -2, $\frac{2}{3}$.

2. $6x^5-17x^4-25x^3+55x^2+39x-18=0.$

OPERATION

$$\begin{array}{cccccc|c}
6x^5 & -17x^4 & -25x^3 & +55x^2 & +39x & -18 & 2 \\
 & -5 & -35 & -15 & 9 & 0 & 3 \\
 & 13 & 4 & -3 & 0 & & -1 \\
 & 7 & -3 & 0 & & &
\end{array}$$

By Art. 348, $$6x^2+7x-3=0,$$

whence $$x^2+\tfrac{7}{6}x=\tfrac{1}{2},$$

and $$x=-\tfrac{7}{12}\pm\tfrac{11}{12}=\tfrac{1}{3}\text{ or }-\tfrac{3}{2}.$$

Hence the roots are 2, 3, -1, $\frac{1}{3}$, $-\frac{3}{2}$.

3. $5x^3-22x^2+15x+18=0.$

4. $6x^3-29x^2-6x+5=0.$

5. $6x^4+5x^3-25x^2-10x+24=0.$

6. $4x^5-28x^4+57x^3-8x^2-67x+30=0.$

366. By Art. 349, produce the equations whose roots are the following, performing the multiplications, as far as practicable, as in Art. 59:

1. $1,\ -3,\ 4.$

2. $1,\ 3,\ -2,\ -4.$

3. $\sqrt{2},\ -\sqrt{2},\ -1,\ 3.$

4. $1,\ 2,\ 2,\ -3,\ 4.$

5. $2,\ 3,\ 4,\ -1,\ -5.$

6. $1,\ 3,\ -2,\ -2,\ -2.$

7. $-3,\ 2+\sqrt{-1},\ 2-\sqrt{-1}.$

8. $\pm\sqrt{-2},\ \pm\sqrt{5}.$

9. $1\pm\sqrt{-2},\ 2\pm\sqrt{-3}.$

10. $\frac{3}{2},\ 2,\ \sqrt{3},\ -\sqrt{3}.$

SOLUTION. $(x-\frac{3}{2})(x-2)(x-\sqrt{3})(x+\sqrt{3})=0,$

$(2x-3)(x-2)(x^2-3)=0,$ and $2x^4-7x^3+21x-18=0.$

QUERY. Why may the denominator of $\dfrac{2x-3}{2}$ be dropped?

11. $2,\ -4,\ \frac{1}{2},\ -\frac{3}{4}.$

12. $1,\ -\frac{1}{2},\ -\frac{2}{3},\ -\frac{3}{4}.$

CHAPTER XVII

SIMULTANEOUS EQUATIONS OF THE SECOND DEGREE WITH TWO UNKNOWN QUANTITIES

367. A Homogeneous Equation is an equation in which all the terms are of the same degree.*

Thus, $3x^2 - 2xy = 4y^2$, $x^2 - 4xy + 5y^2 = 0$, $ax^3 + bx^2y + cxy^2 + dy^3 = 0$, are homogeneous equations.

368. A Symmetrical Equation is an equation in which the unknown quantities may change places without affecting the equality.

Thus,

$3x^2 + 3y^2 - x - y = 5$, $2x^2 + 2y^2 - 3xy = 7$, $ax^2 + ay^2 + bxy + cx + cy = d$, are symmetrical equations.

369. Theorem. *The solution of two equations of the second degree with two unknown quantities requires, in general, the solution of a biquadratic.*

DEM. Two *general equations* of the second degree with two unknown quantities (such equations must provide for all terms that can possibly occur) have the forms

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \tag{1}$$

$$a'x^2 + b'xy + c'y^2 + d'x + e'y + f' = 0. \tag{2}$$

From (1) $$x = -\frac{by + d}{2a} \pm \sqrt{\frac{(by + d)^2}{4a^2} - \frac{cy^2 + ey + f}{a}}.$$

The substitution of this in (2) will give terms in y^2 and a radical of the second degree. Then the rationalization of this equation will require the squaring of a polynomial containing y^2, and the resulting equation will be of the fourth degree.

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* Some writers apply the name to equations in which all the terms except an absolute term are of the same degree.

370. Although the elimination of one of the two unknown quantities from two equations of the second degree results, in general, as shown above, in a biquadratic, there are many cases in which, owing to the absence of some of the terms, the solution may be effected by the use of a quadratic, or by the use of two simple equations with two unknown quantities. The most useful of the various methods are here given.

CASE I

371. *When one of the equations is of the first degree.*

RULE. *Find from the simple equation the value of one of the unknown quantities in terms of the other and known quantities. Substitute this in the other equation and solve in the usual way the resulting quadratic.*

DEM. The only feature of this rule needing proof is that the equation resulting from eliminating one of the unknown quantities is a quadratic.

The general form of an equation of the second degree with two unknown quantities is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

The general form of an equation of the first degree, with two unknown quantities, is

$$mx + ny + p = 0.$$

From the latter, $\quad x = \dfrac{-ny - p}{m},$

which, substituted in the former, gives no term containing a higher power of y than the second.

EXAMPLES XCIV

Solve the following:

1. $\begin{cases} 7x^2 - 8xy = 159, \\ 5x + 2y = 7. \end{cases}$

2. $\begin{cases} x^2 - 2xy - y^2 = 1, \\ x + y = 2. \end{cases}$

3. $\begin{cases} x + y = 4, \\ \dfrac{1}{x} + \dfrac{1}{y} = 1. \end{cases}$

4. $\begin{cases} x^2 + 4xy + 2x + 3y = 17, \\ 2x - y = 0. \end{cases}$

5. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{3}{10}, \\ \dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{1}{20}. \end{cases}$

6. $\begin{cases} \dfrac{a}{x}+\dfrac{b}{y}=2, \\ \dfrac{a^2}{x^2}+\dfrac{b^2}{y^2}=2. \end{cases}$

Although the last two examples contain no equation of the first degree with reference to x and y, the first equation of each is of the first degree with reference to the reciprocals of x and y, and the second equation of each contains the squares of these reciprocals. We may, therefore, regard $\frac{1}{x}$ $\left(\text{or in the 6th } \frac{a}{x}\right)$ as the unknown quantity, and solve accordingly.

Case II

372. *When each equation has only one term of the second degree and these terms are similar.*

Rule. *Eliminate the terms of the second degree, and combine the resulting equation of the first degree with either of the original equations, as in Case I.*

EXAMPLES XCV

Solve the following:

1. $\begin{cases} y^2+3x-4y=6, \\ y^2-x=14. \end{cases}$

2. $\begin{cases} 6x^2+9x-2y=15, \\ 3x^2+6x-4y=18. \end{cases}$

3. $\begin{cases} 2xy-2x+4y=4, \\ xy+x+6y=6. \end{cases}$

4. $\begin{cases} 3xy+6x-3y=10, \\ 6xy+2x-y=10. \end{cases}$

Case III

373. *When one of the equations is homogeneous.*

Rule. *Find from the homogeneous equation the value of one unknown quantity in terms of the other and known quantities. Substitute this value in the other equation and solve the resulting quadratic.*

Dem. The general form of a homogeneous equation of the second degree with two unknown quantities being

$$ax^2+bxy+cy^2=0,$$

we have
$$x=-\frac{by}{2a}\pm\frac{y}{2a}\sqrt{b^2-4ac},$$

or
$$x=\frac{y}{2a}(-b\pm\sqrt{b^2-4ac}).$$

As this is of the first degree with reference to y, no term higher than the second degree can result from substituting this value of x in an equation of the second degree.

EXAMPLES XCVI

Solve the following:

1. $\begin{cases} x^2 + y^2 + 2x = 12, \\ 3x^2 + 2xy - y^2 = 0. \end{cases}$

2. $\begin{cases} x^2 + 5xy + 6y^2 = 180, \\ x^2 + xy - 6y^2 = 0. \end{cases}$

3. $\begin{cases} 2x^2 - 2x + y = 13, \\ x^2 - 4xy + 3y^2 = 0. \end{cases}$

4. $\begin{cases} 8x^2 - 6xy + y^2 = 0, \\ x^2 + y^2 + x = 6. \end{cases}$

Case IV

374. *When both equations have an absolute term, but are otherwise homogeneous.*

FIRST METHOD

Rule. *Eliminate the absolute terms and then proceed as in Case III.*

Example.
$$\begin{cases} x^2 - xy + y^2 = 21, & (1) \\ 2xy - y^2 = 15. & (2) \end{cases}$$

Solution. Multiplying (1) by 5 and (2) by 7, and subtracting, we have

$$\begin{array}{r} 5x^2 - 5xy + 5y^2 = 105 \\ 14xy - 7y^2 = 105 \\ \hline 5x^2 - 19xy + 12y^2 = 0 \end{array}$$

This solved for x gives

$$x = \tfrac{19}{10}y \pm \tfrac{11}{10}y = 3y \text{ or } \tfrac{4}{5}y. \qquad (3)$$

Substituting the first of these values in (2) and solving, we have

$$y = \pm\sqrt{3}.$$

Substituting these values of y in *that part of* (3) *used in finding them*, we have

$$x = 3y = \pm 3\sqrt{3}.$$

Substituting the other value of x, viz., $\frac{4}{5}y$, in (2) and solving, we have

$$y = \pm 5.$$

Substituting these values of y in *that part of* (3) *used in finding them*, we have

$$x = \tfrac{4}{5}y = \pm 4.$$

It should be noted that the values of x and y occur in pairs, thus:

$$\begin{cases} x = 3\sqrt{3}, \\ y = \sqrt{3}. \end{cases} \quad \begin{cases} x = -3\sqrt{3}, \\ y = -\sqrt{3}. \end{cases} \quad \begin{cases} x = 4, \\ y = 5. \end{cases} \quad \begin{cases} x = -4, \\ y = -5. \end{cases}$$

SECOND METHOD

375. RULE. *Find from one of the given equations the value of one of the unknown quantities in terms of the other and known quantities. Substitute this value in the other equation and solve in the usual way the resulting equation, which will always have the quadratic form* (Art. 359).

EXAMPLE.
$$\begin{cases} x^2 - xy + y^2 = 21, & (1) \\ 2\,xy - y^2 = 15. & (2) \end{cases}$$

SOLUTION. From (2)
$$x = \frac{15 + y^2}{2\,y}. \qquad (3)$$

Substituting this value of x in (1) and reducing, we have

$$y^4 - 28\,y^2 = -75,$$

whence
$$y^2 = 14 \pm 11 = 25 \text{ or } 3,$$

and
$$y = \pm 5 \text{ or } \pm\sqrt{3}.$$

These values of y substituted in (3) give

$$x = \pm 4 \text{ or } \pm 3\sqrt{3}.$$

THIRD METHOD

376. RULE. *Assume $x = vy$, and substitute in both equations. By elimination form an equation involving only v, and solve for v. Simple substitution will then determine x and y.*

DEM. Taking the most general form of the equations in question, we have

$$ax^2 + bxy + cy^2 = m,$$

$$dx^2 + exy + fy^2 = n.$$

Now if
$$x = vy, \qquad (1)$$

where v is simply the ratio of x to y,

$$av^2y^2 + bvy^2 + cy^2 = m,$$

$$dv^2y^2 + evy^2 + fy^2 = n;$$

whence
$$y^2 = \frac{m}{av^2 + bv + c} = \frac{n}{dv^2 + ev + f}. \qquad (2)$$

As this equation is of the second degree with reference to v, it can always be solved.

When the value of v is substituted in either of the equations (2), y becomes known; and when v and y are substituted in (1), x becomes known.

EXAMPLE. $$\begin{cases} x^2 - xy + y^2 = 21, \\ 2xy - y^2 = 15. \end{cases}$$

SOLUTION. Assuming $$x = vy, \tag{1}$$

and substituting in both equations, we have

$$y^2 = \frac{21}{v^2 - v + 1} = \frac{15}{2v - 1}, \tag{2}$$

whence $$v = 3 \text{ or } \tfrac{4}{5}.$$

Substituting the first value of v in (2),

$$y^2 = \frac{15}{2v - 1} = 3,$$

whence $$y = \pm\sqrt{3}.$$

Substituting in (1), $$x = vy = \pm 3\sqrt{3}.$$

Substituting the second value of v in (2),

$$y^2 = \frac{15}{2v - 1} = 25,$$

whence $$y = \pm 5.$$

Substituting in (1), $$x = vy = \pm 4.$$

EXAMPLES XCVII

Solve the following:

1. $\begin{cases} x^2 + xy = 15, \\ xy - y^2 = 2. \end{cases}$

2. $\begin{cases} x^2 + xy + 4y^2 = 6, \\ 3x^2 + 8y^2 = 14. \end{cases}$

3. $\begin{cases} x^2 + xy + 2y^2 = 74, \\ 2x^2 + 2xy + y^2 = 73. \end{cases}$

4. $\begin{cases} x^2 + y^2 + 1 = 3xy, \\ 2(xy + 4) = 3y^2. \end{cases}$

5. $\begin{cases} x^2 + xy + y^2 = 52, \\ xy - x^2 = 8. \end{cases}$

6. $\begin{cases} x^2 - 2xy - y^2 = 31, \\ \frac{1}{2}x^2 + 2xy - y^2 = 101. \end{cases}$

377. Many equations falling under Case IV., viz., *equations having an absolute term, but otherwise homogeneous*, admit of shorter solutions than by any of the three methods given above.

1st. When any multiple of one equation added to or subtracted from the other gives a perfect square.

RULE. *If two simple equations are thus obtained, finish the solution by any of the methods of elimination. If but one simple equation is obtained, finish the solution as in Case I.*

EXAMPLE. $$\begin{cases} x^2 + y^2 = 65, & (1) \\ xy = 28. & (2) \end{cases}$$

SOLUTION. Adding twice (2) to (1), we have

$$x^2 + 2\,xy + y^2 = 121,$$

whence $$x + y = \pm 11. \tag{3}$$

Subtracting twice (2) from (1), we have

$$x^2 - 2\,xy + y^2 = 9,$$

whence $$x - y = \pm 3. \tag{4}$$

The combination of (3) and (4) gives

$$x = \pm 7 \text{ or } \pm 4,$$

$$y = \pm 4 \text{ or } \pm 7.$$

The same process is often applicable to equations above the second degree.

EXAMPLE. $$\begin{cases} x^4 + 3\,x^2y^2 = 28, & (1) \\ x^2y^2 + 4\,y^4 = 8. & (2) \end{cases}$$

SOLUTION. The addition of (1) and (2) gives

$$x^4 + 4\,x^2y^2 + 4\,y^2 = 36,$$

whence $$x^2 + y^2 = \pm 6,$$

and $$y^2 = \pm 6 - x^2. \tag{3}$$

Substituting this in (1) and reducing, we have

$$x^4 \mp 9\,x^2 = -14,$$

whence $$x^2 = \pm \tfrac{9}{2} \pm \tfrac{5}{2} = \pm 7 \text{ or } \pm 4,$$

and $$x = \pm\sqrt{7} \text{ or } \pm\sqrt{-7}, \text{ or } \pm 2 \text{ or } \pm 2\sqrt{-1}.$$

By substituting these values of x in (3), y may be found.

2d. When the first members of the two equations have a common factor containing the unknown quantity.

RULE. *Divide one equation by the other, canceling the common factor. Then proceed as in Case I.*

EXAMPLE. $\begin{cases} x^2 + xy = 35, & (1) \\ xy + y^2 = 14. & (2) \end{cases}$

SOLUTION. The equations may be put in the forms

$$x(x+y) = 35, \tag{3}$$

$$y(x+y) = 14. \tag{4}$$

Dividing (3) by (4), we have

$$\frac{x}{y} = \frac{35}{14},$$

whence $$x = \frac{35\,y}{14}. \tag{5}$$

Substituting in (2) and solving,

$$y = \pm 2.$$

Substituting these values of y in (5),

$$x = \pm 5.$$

The same process is applicable to equations above the second degree whose first members have a common factor containing the unknown quantity.

EXAMPLE. $\begin{cases} x^3 + y^3 = 28, & (1) \\ x + y = 4. & (2) \end{cases}$

SOLUTION. Dividing (1) by (2), we have

$$x^2 - xy + y^2 = 7. \tag{3}$$

Proceeding with (2) and (3) as in Case I., we find

$$x = 1 \text{ or } 3,$$

$$y = 3 \text{ or } 1.$$

EXAMPLES XCVIII

Solve the following by Art. 377:

1. $\begin{cases} x^2 - y^2 = 12, \\ xy + y^2 = 12. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 74, \\ xy = 35. \end{cases}$

3. $\begin{cases} x^2y + xy^2 = 30, \\ x + y = 5. \end{cases}$

4. $\begin{cases} x^3 + y^3 = 91, \\ x + y = 1. \end{cases}$

5. $\begin{cases} x^2 - 4\,y^2 = 9, \\ xy + 2\,y^2 = 3. \end{cases}$

6. $\begin{cases} x^2 + 3\,xy = 10, \\ xy + 4\,y^2 = 6. \end{cases}$

7. $\begin{cases} x^3 - y^3 = 875, \\ x^2 + xy + y^2 = 175. \end{cases}$

8. $\begin{cases} x^4 + x^2y^2 + y^4 = 133, \\ x^2 - xy + y^2 = 7. \end{cases}$

Case V

378. *When the equations are symmetrical, and not included in any of the preceding cases.*

Rule. *Assume $x = u + v$ and $y = u - v$, and substitute in both equations. Then reduce and eliminate v.*

Dem. Since x and y are by hypothesis involved alike (Art. 368), $u + v$ and $u - v$ will be involved alike. Hence for every plus term containing an odd power of v there will be an equal negative term. Therefore only even powers of v will remain, and v can be eliminated.

The method is not limited to equations of the second degree.

EXAMPLES XCIX

Solve the following:

1. $$\begin{cases} xy(x+y) = 30, & (1) \\ x^3 + y^3 = 35. & (2) \end{cases}$$

Solution. Assume $$x = u + v, \quad (3)$$
and $$y = u - v. \quad (4)$$

Substituting these values in (1) and (2) and reducing, we have

$$u^3 - uv^2 = 15, \quad (5)$$
and $$2u^3 + 6uv^2 = 35. \quad (6)$$

Adding 6 times (5) to (6), we have

$$8u^3 = 125,$$
whence $$u = \tfrac{5}{2}.$$

Substituting this value of u in (5) and solving, we obtain

$$v = \pm \tfrac{1}{2}.$$

Substituting these values of u and v in (3) and (4), we obtain

$$x = \tfrac{5}{2} \pm \tfrac{1}{2} = 3 \text{ or } 2,$$
and $$y = \tfrac{5}{2} \mp \tfrac{1}{2} = 2 \text{ or } 3.$$

In this example a shorter process is to add 3 times the first equation to the second, extract the cube root of the result, giving $x + y = 5$, and substitute this value in the first equation, giving $xy = 6$. Then the equations $x + y = 5$ and $xy = 6$ are easily solved.

2. $\begin{cases} 4(x+y) = 3xy, \\ x^2 + y^2 + x + y = 26. \end{cases}$

3. $\begin{cases} x^4 - 9x^2y^2 + y^4 = 1, \\ x + y = 4. \end{cases}$

4. $\begin{cases} 2x^2 + 2y^2 = 5xy, \\ 4(x+y) = xy. \end{cases}$

5. $\begin{cases} x^2 + y^2 - x - y = 14, \\ xy + x + y = 14. \end{cases}$

MISCELLANEOUS EXAMPLES

EXAMPLES C

Solve the following:

1. $\begin{cases} x^2 + xy = 12, \\ xy + y^2 = 2. \end{cases}$

2. $\begin{cases} 3xy - x - 5y = 8, \\ xy + x - 3y = 4. \end{cases}$

3. $\begin{cases} x^2 - y^2 = 45, \\ x - y = 3. \end{cases}$

4. $\begin{cases} x^2 + 3xy = 28, \\ xy + 4y^2 = 8. \end{cases}$

5. $\begin{cases} x^2 - 6xy + y^2 - 2x + 10y = -12, \\ x^2 + 6xy - 7y^2 = 0. \end{cases}$

6. $\begin{cases} x^2 - y^2 = 60, \\ xy = 16. \end{cases}$

7. $\begin{cases} x^2 + y^2 + x + y = 32, \\ xy = 12. \end{cases}$

8. $\begin{cases} x^2 + xy = 12, \\ xy - 2y^2 = 1. \end{cases}$

9. $\begin{cases} x^2 + y^2 + x + y = 18, \\ xy = 6. \end{cases}$

10. $\begin{cases} x^4 + y^4 = 97, \\ x + y = 5. \end{cases}$

11. $\begin{cases} x + y + \sqrt{x+y} = 6, \\ x^2 + y^2 = 10. \end{cases}$

12. $\begin{cases} \dfrac{a^2}{x^2} + \dfrac{b^2}{y^2} = 10, \\ \dfrac{ab}{xy} = 3. \end{cases}$

13. $\begin{cases} x^2 + xy = 15, \\ xy - y^2 = 2. \end{cases}$

14. $\begin{cases} x + \sqrt{xy} + y = 19, \\ x^2 + xy + y^2 = 133. \end{cases}$

15. $\begin{cases} x^3 - y^3 = 26, \\ x^2y - xy^2 = 6. \end{cases}$

16. $\begin{cases} 8x^3 - 27y^3 = 271, \\ 2x - 3y = 1. \end{cases}$

17. $\begin{cases} 3x^2 + xy - 2y^2 = 16y, \\ x^2 - 2y^2 = 4y. \end{cases}$

18. $\begin{cases} x^2 + 3xy + 2y^2 = 63, \\ 3x^2 + 8xy + 4y^2 = 171. \end{cases}$

19. $\begin{cases} x^3 + y^3 = 152, \\ x^2 - xy + y^2 = 19. \end{cases}$

20. $\begin{cases} x^2 + y^2 + 3xy - 4(x+y) = -3, \\ xy + 2(x + y) = 5. \end{cases}$

PROBLEMS LEADING TO SIMULTANEOUS EQUATIONS OF THE SECOND DEGREE WITH TWO UNKNOWN QUANTITIES

EXAMPLES CI

1. A certain rectangle contains 300 square feet; a second rectangle is 8 feet shorter and 10 feet broader, and also contains 300 square feet. Find the dimensions of the first rectangle.

2. The area of a rectangular field is 300 square rods, and the length of its diagonal is 25 rods. Find the sides.

3. A man bought some horses for \$1250. At another time he bought 3 more than before, paying \$25 less apiece, and they cost him \$1300. How many horses did he buy the first time, and at what price?

4. The fore wheel of a buggy makes 6 revolutions more than the hind wheel in going 120 yards; but the fore wheel of a coach, each of whose wheels is larger in circumference by 1 yard respectively, makes only 4 revolutions more than the hind wheel in going the same distance. What is the circumference of each buggy wheel?

5. In walking to the summit of a mountain a man's rate during the second half of the distance is $\frac{1}{2}$ mile per hour less than during the first half, and he reaches the summit in $5\frac{1}{2}$ hours. He descends in $3\frac{3}{4}$ hours at a uniform rate, which is 1 mile per hour more than his rate during the first half of the ascent. Find the distance to the summit and the rates of walking.

6. Two trains start at the same time from two places, A and B, 168 miles apart, and travel toward each other. They pass in 1 hr. 52 min., and the first reaches B $\frac{1}{2}$ hour before the second reaches A. Find the speed of each train.

7. A crew, rowing at half their usual speed, row 3 miles down stream and back in 2 hr. 40 min. At full speed they can go over the same course in 1 hr. 4 min. Find the rate of the crew and of the current.

8. A courier, riding at a uniform rate, left a station. Five hours afterward a second followed, riding 3 miles an hour faster. Two hours after the second a third started at the rate of 10 miles an hour. They all reached their destination at the same time. Find the distance; also the rate of the first.

9. A and B are two towns situated 18 miles apart on the same bank of a river. A man goes from A up to B in 4 hours, rowing the first half of the distance and walking the second half. In returning he walks the first half at the same rate as before, but the stream being with him, he rows $1\frac{1}{2}$ miles per hour faster than in going, and covers the whole distance in $3\frac{1}{2}$ hours. Find the rates of rowing and walking.

10. A man arrives at the railroad station nearest his home $1\frac{1}{2}$ hours before the time at which he has ordered his carriage to meet him. He sets out at once to walk at the rate of 4 miles an hour, and meeting his carriage when it has traveled 8 miles, reaches home 1 hour earlier than he had originally expected. How far is his home from the station, and at what rate was his carriage driven?

CHAPTER XVIII

THEORY OF FUNCTIONS

SECTION I—MAXIMA AND MINIMA OF FUNCTIONS

379. An Arbitrary Constant is a constant to which any value may be assigned, but which maintains the same value throughout the same operation or discussion.

An Absolute Constant is a constant which admits of no change.

Thus, in the formula for the area of a circle, πr^2, r is an arbitrary constant, since it may have any value we choose to give it; but π, always having the same value, viz., 3.14159 approximately, is an absolute constant. When we assign to r any particular value, as 5, the radius becomes an absolute constant.

Arbitrary constants are represented by the leading letters of the alphabet, and absolute constants by figures or by letters which always stand for the same numbers.

380. A Variable is a quantity which may have in the same operation or discussion any value within the limits determined by the conditions.

Thus, if $y = \sqrt{25 - x^2}$, and this is the only required relation between x and y, we may give to x any values whatever between -5 and $+5$ and find corresponding values of y. Hence x and y are variables. If x is made less than -5 or greater than $+5$, y is imaginary. Hence the limits of x are -5 and $+5$.

Variables are represented by the final letters of the alphabet.

381. Constants and variables are not the same as known and unknown quantities, although the notation is the same. In the simultaneous equations $5x + 2y = 7$ and $7x^2 - 8xy = 159$, x and y have two, and only two, values each, and hence they are constants; but if x and y are required to fulfill only the one condition expressed in the first equation, $5x + 2y = 7$, we can give to one of them any value we please, and can find for the other such value

as will make the equation true. In this case x and y are variables.

382. A Function of a Variable is any expression which depends upon that variable for its value.

Thus, $\sqrt{25-x^2}$ is a function of x, inasmuch as it changes when x changes. Any expression containing x is a function of x. If we have $y=\sqrt{25-x^2}$, we say y is a function of x.

383. A Function of Two or More Variables is any expression which depends upon those variables for its value.

Thus, interest on a loan of money is a function of the principal, the rate of interest, and the time; the volume of a cone is a function of the radius of the base and the altitude; the distance passed over by a body moving from rest with a uniformly accelerated velocity ($D=\frac{1}{2}ft^2$) is a function of the acceleration and the time.

384. Notation. A function of x is represented by $f(x)$. This is employed not only to represent *any* function of a single variable, but also to represent a specified function to avoid repetition of the function itself. In this expression f is not a factor, but simply an abbreviation of the word *function.*

When different functions of the same variable are brought into the same discussion, the notation

$$f(x),\ f'(x),\ f''(x),\ f_1(x),\ f_2(x),\ \phi(x),\ \phi'(x),\ \text{etc.},$$

is employed. These are read, "f function of x," "f' function of x," "f'' function of x," "f_1 function of x," "f_2 function of x," "ϕ function of x," "ϕ' function of x," etc.

To represent the same function of different variables, the notation $f(x)$, $f(y)$, $f(z)$, $f(-x)$, etc., is employed.

To indicate that a constant has been substituted for the variable in a function, the notation $f(a)$, $f(2)$, $f(0)$, etc., is employed.

Thus, if $\quad f(x)=2x^3-3x^2-4x+5,$

then

$$\begin{aligned}
f(y)&=2y^3-3y^2-4y+5,\\
f(z)&=2z^3-3z^2-4z+5,\\
f(-x)&=-2x^3-3x^2+4x+5,\\
f(a)&=2a^3-3a^2-4a+5,\\
f(2)&=2\cdot 2^3-3\cdot 2^2-4\cdot 2+5=1,\\
f(0)&=5.
\end{aligned}$$

To represent a function of several variables, the notation $f(x, y)$, $f(x, y, z)$, etc., is employed. These are read, "function of x and y," "function of x, y, and z," etc.

In the expression $F(x, y)$, x and y are understood to be independent of each other; but if the expression occurs as part of an equation, as $f(x, y) = 0$, x and y are dependent. If the equation can be solved for one of the variables, say y, we shall have

$$y = \phi(x).$$

385. **An Increasing Function** is a function that increases when its variable increases, and decreases when its variable decreases.

A Decreasing Function is a function that decreases when its variable increases, and increases when its variable decreases.

Thus, $y = x^3$, $y = x^5 + x$, $y = mx + b$, are increasing functions;

$$y = -x^3, \quad y = \frac{1}{x}, \quad y = a - x,$$

are decreasing functions; while $y = x^2$ is an increasing function for positive values of x, and a decreasing function for negative values of x.

386. **A Rational Integral Function** of x is a function in the form

$$ax^n + bx^{n-1} + cx^{n-2} + \cdots l,$$

in which all the exponents are positive integers.

387. Functions are **linear, quadratic, cubic,** etc., according as they are of the first, the second, the third, etc., degree.

388. The notation $\Big]_m$ signifies that m is to be substituted for the variable in the function after which it is written.

Thus, $\left.\dfrac{x^2 - 4x + 3}{x^2 - x - 6}\right]_3$ means the value of the function when 3 is substituted for x.

389. **A Maximum Value of a Function** is that which is greater than the immediately preceding and succeeding values.

A Minimum Value of a Function is that which is less than the immediately preceding and succeeding values.

Thus, for larger and larger values of x in the function $6x^2 - x^3$, we have the following results:

Values of x,	-3,	-2,	-1,	0,	1,	2,	3,	4,	5,	6,	7, etc.
Values of $f(x)$,	81,	32,	7,	0,	5,	16,	27,	32,	25,	0,	-49, etc.

It is seen that for increasing values of x, $f(x)$ at first decreases until it reaches 0 when $x=0$, then increases until it reaches 32 when $x=4$, and then decreases for all larger values of x. Hence 0 is a minimum value of $f(x)$, and 32 is a maximum value of $f(x)$. We say, also, that $x=0$ renders $f(x)$ a minimum, and $x=4$ renders $f(x)$ a maximum.

390. NOTE. The greatest value of a function is not necessarily a maximum, and the least value is not necessarily a minimum, in the mathematical sense of these words. For example, the least possible value of the function $5+\sqrt{x-3}$ (using only the positive sign of the radical) is 5, and this is when $x=3$. This, however, is not a minimum, since it is not less than both the preceding and the succeeding values — indeed, there is no preceding value, for if x is made less than 3, the function is imaginary.

Again, a minimum of a function may be larger than a maximum of the same function. Thus, let the function $\sqrt{x^2+9}$ be represented by y, giving $\sqrt{x^2+9}=y$, or $x=\pm\sqrt{y^2-9}$. In this form it is seen that y can have no values between -3 and $+3$, as these would render x imaginary, but can have all values beyond these limits. Hence -3, being the largest of all the negative values of y, is a maximum, and $+3$, being the smallest of all the positive values of y, is a minimum. In this case, then, the minimum value of the function is larger than the maximum.

391. A full discussion of the subject of maxima and minima of functions requires the aid of the Differential Calculus; but the maxima and minima of many functions may be determined by methods that are purely algebraic, and the same methods apply to numerous interesting and important practical problems.

CASE I — QUADRATIC FUNCTIONS

392. Theorem. *1st. A quadratic function, ax^2+bx+c, has a maximum if a is $-$, and a minimum if a is $+$.*

2d. The value of x which renders the function a maximum or a minimum is $-\dfrac{b}{2a}$.

3d. The maximum or minimum value of the function is $c-\dfrac{b^2}{4a}$.

DEM. 1st. Let $ax^2+bx+c=y$.
Solving for x, we have

$$x=-\frac{b}{2a}\pm\frac{1}{2a}\sqrt{b^2+4ay-4ac}. \tag{1}$$

If a is $-$, the term $4ay$ will have the opposite sign from y; hence y can decrease without limit ($4ay$ becomes $+$ when y decreases below 0), but can increase only to the value beyond which the quantity under the radical sign would become $-$, making x imaginary. This limit is a maximum value of y.

If a is $+$, the term $4ay$ will have the same sign as y; hence y can increase without limit, but can decrease only to the value beyond which the quantity under the radical sign would become $-$, making x imaginary. This limit is a minimum value of y.

2d. In either case the limit, which makes the function a maximum or a minimum, is where

$$b^2+4ay-4ac=0, \tag{2}$$

which gives

$$x=-\frac{b}{2a}.$$

3d. The maximum or minimum of the function may be found either by substituting this value of the variable in the given function, or by solving (2) for y, which gives

$$y=c-\frac{b^2}{4a}.$$

393. Cor. *The value of the variable which renders a pure quadratic function, ax^2+c, a maximum or a minimum is* 0, *and the maximum or minimum value of the function is* c.

This is because $x=-\dfrac{b}{2a}=\dfrac{0}{2a}=0$, and $y=c-\dfrac{b^2}{4a}=c-\dfrac{0}{4a}=c$.

EXAMPLES CII

Examine the following functions for maxima and minima values:

1. $x^2-4x+10$.

SOLUTION. By 1st and 2d parts of the theorem,

$$x=-\frac{b}{2a}=-\frac{-4}{2}=2$$

renders $f(x)$ a minimum, since a is $+$.

By the 3d part of the theorem,

$$f(x) \text{ at a min.} = c - \frac{b^2}{4a} = 10 - \frac{16}{4} = 6.$$

Or
$$f(x) \text{ at a min.} = x^2 - 4x + 10]_2 = 6.$$

2. $-2x^2 + 6x - 5$.

SOLUTION.
$$x = -\frac{b}{2a} = -\frac{6}{-4} = 1\tfrac{1}{2}$$

renders $f(x)$ a maximum, since a is $-$.

$$f(x) \text{ at a max.} = -2x^2 + 6x - 5]_{1\frac{1}{2}} = -\tfrac{1}{2}.$$

3. $6x - x^2$.
4. $x^2 - 8x + 19$.
5. $3x^2 + 12x + 12$.
6. $7 + 8x - 2x^2$.
7. $5x^2 - 20x + 20$.
8. $16x - 2x^2 + 18$.
9. $6x^2 + 11$.
10. $-3x^2 - 3x + 7$.

CASE II — RECIPROCALS OF QUADRATIC FUNCTIONS

394. Theorem. *The value of the variable which renders any function a maximum or a minimum, renders the reciprocal of that function a minimum or a maximum.*

DEM. This is seen from the fact that when $f(x)$ increases or decreases, $\dfrac{1}{f(x)}$ decreases or increases.

395. SCH. The reciprocal of a rational integral function approaches 0 as a limit as the variable approaches $\pm\infty$. This limit is not a minimum or a maximum in a mathematical sense (Art. 389), but is merely the least or the greatest value possible.

EXAMPLES CIII

Examine the following functions for maxima and minima values:

1. $\dfrac{1}{x^2 - 2x - 1}$.

SOLUTION. By Art. 392,

$$x = -\frac{b}{2a} = -\frac{-2}{2} = 1$$

renders the denominator a minimum. Hence (Art. 394) $x=1$ renders the reciprocal function a maximum.

$$f(x) \text{ at a max.} = \frac{1}{x^2-2x-1}\bigg]_1 = -\frac{1}{2}.$$

2. $\dfrac{1}{4-6x-3x^2}$.

3. $\dfrac{-57}{2x^2+6x-5}$.

4. $\dfrac{18m}{x^2-8x+7}$.

5. $\dfrac{n-4}{3+20x-5x^2}$.

Case III — Algebraic Functions of Any Form

396. Prob. *To examine for maxima or minima values algebraic functions of any form.*

Rule. *1st. Place the function equal to y, and solve for the variable in terms of y.*

2d. If the result does not involve a radical of even degree, the function has neither a maximum nor a minimum.

3d. If the result involves a radical of even degree, place this radical part equal to 0 and solve for y. Each resulting real value of y will be a maximum or a minimum according as an increase or a decrease would give an imaginary value of the variable.

Dem. If after expressing the variable in terms of y there is no radical of even degree, it is evident that y, which is the function, can increase or decrease without limit, every value of y giving a real value of the variable.

But if there is a radical of even degree, and the quantity under the radical sign is capable of becoming negative for any value of y, y can increase or decrease only to the limit beyond which this quantity would become negative, making the variable imaginary. This limit, which is reached where the radical part equals 0, is a maximum or a minimum according as an increase or a decrease would give an imaginary result.

397. Sch. 1. The method has its limitations in our inability to solve equations of all forms.

398. Sch. 2. Case III includes Cases I and II, but the special theorems of those cases give the result with less labor.

EXAMPLES CIV

Examine for maxima and minima values the following functions:

1. $a(x-b)^4+c$.

SOLUTION. Let $$a(x-b)^4+c=y.$$

Then $$(x-b)^4=\frac{y-c}{a},$$

$$x-b=\pm\sqrt[4]{\frac{y-c}{a}},$$

and $$x=b\pm\sqrt{\frac{y-c}{a}}.$$

In this form it is seen that y can increase without limit, but can decrease only to $y=c$, which is, therefore, a minimum value. This makes the radical part 0, and leaves $x=b$.

2. $b+(x-a)^3$.

SOLUTION. Let $$b+(x-a)^3=y.$$

Then $$x=a+\sqrt[3]{y-b}.$$

In this form it is seen that y can increase and decrease without limit. Therefore the function has neither a maximum nor a minimum.

In the first example, if y is made less than c, we have the 4th root of a negative quantity, which is imaginary; but in this example no value of y will give an imaginary result, as an odd instead of an even root is involved.

3. $\dfrac{x^2-2x+13}{4x-12}$.

SOLUTION. Let $$\frac{x^2-2x+13}{4x-12}=y.$$

Then $$x^2-(2+4y)x=-12y-13,$$

and $$x=1+2y\pm\sqrt{4y^2-8y-12}.$$

By the 3d part of the rule, for a maximum or minimum

$$4y^2-8y-12=0,$$

whence $$y=3 \text{ or } -1.$$

Any values of y between 3 and -1 give imaginary values for x. From a large value y can diminish to 3, and from a small value (a negative value) y can increase to -1. Hence 3 is a minimum value of the function and -1 is a maximum. [See Art. 390.]

The corresponding values of x are

$$x = 1 + 2y = 7 \text{ and } -1.$$

4. $\dfrac{3x^2 - 6}{2x - x^2}$.

SOLUTION. Let $$\frac{3x^2 - 6}{2x - x^2} = y.$$

Then $$x^2 - \frac{2y}{3+y}x = \frac{6}{3+y},$$

and $$x = \frac{y}{3+y} \pm \frac{1}{3+y}\sqrt{y^2 + 6y + 18}.$$

For a maximum or a minimum (3d part of rule),

$$y^2 + 6y + 18 = 0,$$

whence $$y = -3 \pm 3\sqrt{-1}.$$

This imaginary result shows that our supposition that there is a limit beyond which y cannot go (a limit beyond which the quantity under the radical sign would become $-$, giving an imaginary value of x) is absurd (Art. 227). Hence the function has neither a maximum nor a minimum.

5. $\dfrac{1 - 3x}{x^2 - 2x}$.

SOLUTION. Let $$\frac{1 - 3x}{x^2 - 2x} = y.$$

Then $$x = \frac{2y - 3}{2} \pm \tfrac{1}{2}\sqrt{(2y - 3)^2 + 4}.$$

As y can increase and decrease without limit, the function has neither a maximum nor a minimum. This may also be shown by placing the radical part equal to 0, which gives an imaginary result.

6. $x^4 - 6x^2 - 7$.

SOLUTION. Let $$x^4 - 6x^2 - 7 = y.$$

Then $$x^2 = 3 \pm \sqrt{y + 16},$$

and $$x = \pm\sqrt{3 \pm \sqrt{y + 16}}.$$

In this form it is seen that y cannot be less than -16. This is, therefore, a minimum, and gives $x = \pm\sqrt{3}$.

Again, when the minus sign of $\sqrt{y + 16}$ is used, the largest value of y is that which makes $\sqrt{y + 16} = 3$, which gives $y = -7$, a maximum, and $x = 0$.

7. $b+(x-a)^{\frac{3}{2}}$.

SOLUTION. Let $$b+(x-a)^{\frac{3}{2}}=y.$$

Then $$x=a+\sqrt[3]{(y-b)^2}.$$

By 2d part of the rule, the function has neither a maximum nor a minimum.

8. $\dfrac{x-2}{3-x^2}$.

9. $b+(x-a)^{\frac{4}{3}}$.

10. $\dfrac{2x-3}{x^2}$.

11. $\dfrac{2x-5}{x^2}$.

12. $\dfrac{x^2+2x+1}{x^2+2x+7}$.

13. $(2ax-x^2)^{\frac{1}{2}}$.

14. $2x^4-16x^2+44$.

15. $\dfrac{x+1}{x^2+x+1}$.

16. $b-(x-a)^{\frac{8}{5}}$.

17. $3x^4-12x^2-15$.

399. If the maxima and minima values of each of two variables involved in an equation are required, it is evident that we may solve for each of the variables in turn, and proceed as in Case III.

EXAMPLES CV

Examine for maxima and minima values each of the variables in the following functions:

1. $x^2+8y+14=2y^2+4x$.

SOLUTION. Arranging with reference to x, we have

$$x^2-4x=2y^2-8y-14,$$

whence $$x=2\pm\sqrt{2y^2-8y-10}. \qquad (1)$$

For the limits of y we have

$$2y^2-8y-10=0,$$

whence $$y=5 \text{ or } -1.$$

Since y can have no values between 5 and -1, but can have all values beyond these limits, 5 is a minimum and -1 a maximum value of y.

For both of these values of y equation (1) gives $x=2$.

By solving for y in terms of x it will be found that x has neither a maximum nor a minimum.

2. $x^2+3y^2+36=2x+12y$.

3. $x^2+2y^2=4x+4y+16$.

4. $x^2+y^2=2y+15$.

400. When the theory of maxima and minima is to be applied to a practical problem, the first step is to obtain an algebraic expression for the function that is to be a maximum or a minimum. If this contains but one variable, we proceed as in one of the three cases explained in this section. If it contains two variables, one of them must be eliminated by given relations before applying the process.

PROBLEMS IN MAXIMA AND MINIMA OF FUNCTIONS

EXAMPLES CVI

1. Divide m into two such parts that their product shall be a maximum.

Solution. Letting x and $m - x$ be the two parts, the function to be examined is

$$x(m - x) = -x^2 + mx.$$

By Art. 392, $$x = -\frac{b}{2a} = -\frac{m}{-2} = \frac{m}{2}$$

renders $f(x)$ a maximum, since a is $-$. Therefore the two parts are equal.

2. A farmer having 40 rods of portable fence wishes to inclose with it the largest possible rectangular sheep yard. What must be its dimensions?

Solution. Letting x and y represent the sides, the function to be examined is xy. (1)

Since this contains two variables, one of them must be eliminated. This is done by means of the condition

$$2x + 2y = 40,$$

whence $$y = 20 - x. \qquad (2)$$

Substituting this in (1), the function to be examined becomes

$$x(20 - x) = -x^2 + 20x.$$

By Art. 392, $$x = -\frac{b}{2a} = -\frac{20}{-2} = 10$$

renders $f(x)$ a maximum, since a is $-$.

Substituting this in (2), we have

$$y = 20 - x = 20 - 10 = 10.$$

Hence the yard must be a square.

3. Divide 12 into two such parts that the sum of their squares shall be a maximum.

4. A farmer wishes to fence off a 10-acre field in the form of a rectangle of such dimensions as to require the least amount of fence. Find its dimensions.

5. If it is specified that a purchaser is to have a rectangular plot of ground of such dimensions that 3 times its breadth added to 2 times its length shall equal 96 yards, what is the greatest amount of land he can take?

6. An aqueduct consists of two vertical walls surmounted by a semicylindrical arch, the stone bottom being of the same thickness as the walls. Find the dimensions such that with a given amount of material the capacity shall be a maximum.

Solution. The capacity will be greatest when the area of a cross section is a maximum.

Let x be the height of the side walls, $2y$ the breadth, and p the perimeter, which is constant, since the amount of material is given. Then the function to be examined is

$$2xy + \frac{\pi y^2}{2}. \tag{1}$$

To eliminate x, we have

$$p = 2x + 2y + \pi y,$$

whence
$$x = \frac{p - 2y - \pi y}{2}. \tag{2}$$

Substituting this in (1), we have

$$py - 2y^2 - \pi y^2 + \frac{\pi y^2}{2} = -\frac{4+\pi}{2}y^2 + py.$$

By Art. 392,

$$y = -\frac{b}{2a} = -\frac{p}{-(4+\pi)} = \frac{p}{4+\pi}$$

renders $f(y)$ a maximum, since a is $-$.

Substituting this in (2), we have

$$x = \frac{p}{2} - \frac{p}{4+\pi} - \frac{\pi p}{2(4+\pi)} = \frac{p}{4+\pi}.$$

Hence the height of the side walls equals one half the breadth of the aqueduct.

7. A ship steaming north 12 knots an hour sights another ship 10 knots directly ahead steaming east 9 knots an hour. If each keeps on her course, what will be the least distance between them, and at what time will it occur?

Sug. Let x be the time, and find in terms of x an expression for the square of the distance between the ships. The distance will be a minimum when the square of the distance is a minimum.

8. I have material enough for a stone wall 48 rods in length. I can use for one side any desired portion of a wall already built. What is the largest rectangular area I can inclose, and what are its dimensions?

Show that the relative dimensions would be the same, whatever the amount of wall.

9. Find the side of the least square that can be inscribed in the square whose side is m. Find also the distance of its corners from the corners of the given square.

Sug. Let x and $m-x$ be the distances from a corner of the inscribed square to the adjacent corners of the given square. It will then be found that the function to be examined is

$$2x^2 - 2mx + m^2.$$

10. A circular piece of tin is to be utilized for the bottom of the largest possible rectangular box of given depth. Find the dimensions of the bottom.

Solution. As the depth is fixed, the volume of the box will be a maximum when the area of the bottom is a maximum.

Representing by x and y the half sides of the bottom, the function to be examined is $4xy$. Eliminating y by the relation $x^2 + y^2 = r^2$, we have for the function

$$4x\sqrt{r^2 - x^2}.$$

Proceeding as in Case III, Art. 396, we have

$$u = 4x\sqrt{r^2 - x^2},$$

whence

$$x = \sqrt{\frac{r^2}{2} \pm \frac{1}{4}\sqrt{4r^4 - u^2}}.$$

For a maximum or a minimum

$$4r^4 - u^2 = 0,$$

which gives

$$u = \frac{r^2}{2},$$

and

$$x = \frac{r}{\sqrt{2}}.$$

Substituting this value of x in $x^2 + y^2 = r^2$, we have

$$y = \frac{r}{\sqrt{2}}.$$

Therefore the rectangle is a square.

11. In the last example, if the bottom is to be cut from a semicircle of radius r, what must be its dimensions?

12. Find the greatest right triangle that can be constructed upon a given line as a hypotenuse.

13. A carpenter wishes to make the largest possible rectangular table top from a board 10 feet long, $3\frac{1}{2}$ feet wide at one end, and 1 foot wide at the other. Find the dimensions.

SECTION II — ZERO, INFINITY, AND INDETERMINATE FORMS

401. The symbol 0 is used not only to represent the absence of value, but also to represent a quantity that is less than any assignable value; *i.e.*, it may represent either absolute zero or an infinitesimal.

Although all infinitesimals are not equal, they are all represented by the same symbol.

402. The symbol ∞, called **Infinity**, represents a quantity that is greater than any assignable value.

Although all infinities are not equal, they are all represented by the same symbol.

403. Prob. *To interpret the forms* $\frac{a}{0}$ *and* $\frac{a}{\infty}$.

Solution. If in the fraction $\frac{a}{x}$, x diminishes while a remains constant, the quotient increases; and finally when x becomes less than any assignable value, the quotient becomes greater than any assignable value. Hence $\frac{a}{0} = \infty$.

Again, if in the fraction $\frac{a}{x}$, x increases while a remains constant, the quotient decreases; and finally, when x becomes greater than any assignable value, the quotient becomes less than any assignable value. Hence $\frac{a}{\infty} = 0$.

404. Prob. *To interpret the forms* $\frac{0}{0}$ *and* $\frac{\infty}{\infty}$, *called indeterminate forms.*

Solution. In the fraction $\frac{x}{y}$, in which the variables are inde-

pendent of each other, if x and y diminish until they become less than any assignable values, giving $\frac{0}{0}$, or increase until they become greater than any assignable values, giving $\frac{\infty}{\infty}$, the fraction is indeterminate, the quotient is any number whatever, and the conditions of the problem from which it results are fulfilled for every value.

But if x and y are dependent and their relation is known, for example, if $y = mx$, giving for the fraction $\frac{x}{mx}$, then however small or however large x may become, the numerator will be $\frac{1}{m}$ of the denominator, and the quotient will be $\frac{1}{m}$. Hence if the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ results from the presence in both terms of a fraction of a factor which reduces to 0 or ∞ for a particular value of the variable, the value of the fraction may be determined by dividing out this factor before evaluating.

EXAMPLES CVII

Find the values of the following:

1. $\left.\frac{x^2+5x-2}{x^2-2x+1}\right]_1.$

SOLUTION. Substituting 1 for x, we have $\frac{4}{0} = \infty$ (Art. 403).

2. $\left.\frac{x^2-5x+6}{x^2+2x-8}\right]_2.$

SOLUTION. Substituting 2 for x, the fraction assumes the form $\frac{0}{0}$. We therefore seek for a factor common to both terms of the fraction.

$$\frac{x^2-5x+6}{x^2+2x-8} = \frac{(x-3)(x-2)}{(x+4)(x-2)} = \left.\frac{x-3}{x+4}\right]_2 = -\frac{1}{6}.$$

3. $\left.\frac{x^2-2x+4}{x^2-4x+4}\right]_2.$

4. $\left.\frac{x+3}{x^2-6x+9}\right]_3.$

5. $\left.\frac{2x^2+x+1}{\frac{x+1}{x-1}}\right]_1.$

6. $\left.\frac{x^3+x^2+x+1}{x^{-1}}\right]_0.$

7. $\left.\frac{x^3+1}{x+1}\right]_{-1}.$

8. $\left.\frac{(2+x)^2-4}{5x}\right]_0.$

9. $\left.\dfrac{1+3x}{3+5x}\right]_\infty.$ **10.** $\left.\dfrac{ax^{-2}+bx^{-1}+c}{dx^{-2}+ex^{-1}+f}\right]_\infty.$

11. $\left.\dfrac{x^3-x^2-x+1}{x^3-3x+2}\right]_1.$ **12.** $\left.\dfrac{x^4-5x^3+6x^2+4x-8}{2x^4-13x^3+30x^2-28x+8}\right]_2.$

SECTION III — DISCUSSION OF FUNCTIONS AND PROBLEMS

405. The Discussion of a Function or a Problem is interpreting it for different values of the literal quantities which enter it. If only arbitrary and absolute constants enter it, the discussion is for different values of the arbitrary constants. If variables and constants enter it, the discussion is for different values of the variables alone.

INTERPRETATION OF NEGATIVE RESULTS

406. When the answer or one of the answers to a problem is negative, it must be reckoned in the opposite direction from that assumed as positive in the statement of the problem. For example, let it be required to find the time when A will be (or was) one and one half times as old as B, A's age now being 28 and B's 20. Suppose we assume that this ratio of their ages will be at some time in the future, say x years from now. By the conditions of the problem we have

$$28+x=\tfrac{3}{2}(20+x),$$

whence $$x=-4.$$

As we assumed time in the future +, and stated our equation in accordance with this assumption, the negative result shows that the time is in the past, viz., 4 years ago.

Had we assumed the event in the past, calling past time +, our equation would have been

$$28-x=\tfrac{3}{2}(20-x),$$

whence $$x=4;$$

i.e., 4 years in the direction we assumed as positive, or 4 years ago.

The first answer obtained, -4, is considered correct in the alge-

braic sense, the negative of future time being past time. To obtain the correct answer in an arithmetical sense, the words *will be* in the enunciation of the problem must be changed to *was*.

If, as in Ex. 1, page 213, the nature of the problem is such, either as enunciated or with a suitable change of words, as *was* for *will be*, *loss* for *gain*, *decrease* for *increase*, etc., as not to permit an interpretation of a negative result, then the negative answer must be rejected.

INTERPRETATION OF IMAGINARY RESULTS

407. As stated in Art. 227, an imaginary answer shows that, arithmetically, the conditions of the problem cannot, even with a change of the wording, be fulfilled. For example, let it be required to divide 12 into two such parts that the sum of their squares shall be 54. Representing the two parts by x and $12 - x$, we have by the conditions

$$x^2 + (12 - x)^2 = 54,$$

whence
$$x = 6 \pm 3\sqrt{-1},$$

and
$$12 - x = 6 \mp 3\sqrt{-1}.$$

These imaginary results show that, arithmetically, 12 cannot be divided into two parts the sum of whose squares is 54. If we apply the test of Art. 392, we find that the sum of the squares of the two parts is a minimum when each of the two parts is half of the number. Hence the smallest value of the sum of the squares is $6^2 + 6^2 = 72$. Nevertheless, algebraically, the conditions of the problem are fulfilled by these results, since

the sum $= 6 \pm 3\sqrt{-1} + 6 \mp 3\sqrt{-1} = 12,$

the sum of the squares $= (6 \pm 3\sqrt{-1})^2 + (6 \mp 3\sqrt{-1})^2 = 54.$

408. In discussing a function which has no connection with a problem, the following features should usually be determined:

1. What values of the variable (or arbitrary constants) give but one value to the function.

2. Between what limits of the variable the function has more than one real value, and whether these values are numerically equal or unequal.

3. Between what limits of the variable the function is imaginary.

4. What value of the variable makes the function a maximum or a minimum, and the value of the function at a maximum or a minimum.

In discussing a problem, the above features and others, including negative and indeterminate results, should be interpreted with reference to the particular problem. Of course if the solution involves an equation of only the first degree, there could be no double, imaginary, maxima or minima values.

EXAMPLES CVIII

Discuss the following:

1. $y^2 = x^4 - x^2$.

DISCUSSION. Solving for y, we have

$$y = \pm x\sqrt{x^2 - 1}.$$

1st. For $x = 1$ or $x = 0$, y has but one value, viz., 0.

2d. For all values of $x < -1$ and $> +1$, y has two real values, numerically equal, but with opposite signs.

3d. For all values of x between -1 and 0, and between 0 and $+1$, y is imaginary.

4th. Of all the negative values of x, -1 is the greatest, a maximum; and of all positive values of x, $+1$ is the least, a minimum.

2. $2ay^2 = x^3 + xy^2$.

DISCUSSION. Solving for y, we have

$$y = \pm \frac{x\sqrt{x}}{\sqrt{2a - x}}.$$

1st. For $x = 0$, y has but one value, viz., 0.

2d. For $x > 0$ and $< 2a$, y has two real values, numerically equal, but with opposite signs.

3d. For $x > 2a$ or < 0, y is imaginary.

4th. $x = 0$ is a minimum, and $x = 2a$ is a maximum.

5th. For $x = 2a$, $y = \infty$.

3. $3x^2 + y^2 + 4xy - 12x - 8y + 21 = 0$.

DISCUSSION. Solving for y, we have

$$y^2 + (4x - 8)y = -3x^2 + 12x - 21,$$

or

$$y = -2x + 4 \pm \sqrt{x^2 - 4x - 5}.$$

1st. When $x^2 - 4x - 5 = 0$, which gives $x = 5$ or -1, y has but one value, viz., -6 for the former and 6 for the latter.

2d. When $x^2 - 4x - 5 > 0$, *i.e.*, when it is positive, y has two real, unequal values. This gives, by solving the inequality, $x > 5$ or < -1.

3d. When $x^2 - 4x - 5 < 0$, *i.e.*, when it is negative, which gives $x < 5$ and > -1, y is imaginary.

4th. Since any values of x between -1 and 5 give imaginary values for y, while any values of $x < -1$ or > 5 give real values for y, -1 is a maximum value of x, and 5 is a minimum value of x.

4. $y = \dfrac{a' - a}{1 + aa'}$.

DISCUSSION. 1st. When $a' > a$, y is $+$, and when $a' < a$, y is $-$.

2d. When $a' = a$, $y = \dfrac{0}{1 + a^2} = 0$.

3d. When $1 + aa' = 0$, or $a' = -\dfrac{1}{a}$, $y = \dfrac{-\frac{1}{a} - a}{0} = \infty$.

5. $xy^2 = 4a^2(2a - x)$. **6.** $(x - 1)y^2 = x^2$.

7. $x^2y^2 = a^2(x^2 + y^2)$. **8.** $x^2y = 4a^2(2a - y)$.

9. $y^3 = a^3 - x^2$. **10.** $y^2 - 2xy + x^2 - 4y + x + 4 = 0$.

11. Divide a into two parts whose product shall be p.

SOLUTION AND DISCUSSION. Representing the two parts by x and $a - x$, we have

$$x(a - x) = p,$$

whence

$$x = \frac{a}{2} \pm \tfrac{1}{2}\sqrt{a^2 - 4p},$$

and

$$a - x = \frac{a}{2} \mp \tfrac{1}{2}\sqrt{a^2 - 4p}.$$

1st. For $4p = a^2$, x has but one value, viz., $\dfrac{a}{2}$.

2d. For $4p < a^2$, x has two real, unequal values.

3d. For $4p > a^2$, x is imaginary, showing an arithmetical inconsistency, viz., naming a product larger than any parts of the given number will give.

4th. From $a^2 - 4p = 0$, we have (Art. 393) p at a maximum $= \dfrac{a^2}{4}$, which gives $x = \dfrac{a}{2}$; *i.e.*, to give a maximum product each of the two parts must be half of the given number.

12. Two couriers, A at rate a and B at rate b, are traveling on an east and west road. At noon A is at M and B at N, c miles apart. Find the time and the place of their being together.

Solution and Discussion. Let x be the number of hours between noon and the instant of their being together. Let time after noon and distance to the east of M be regarded as positive. Then

$$ax = bx + c,$$

whence $$x = \frac{c}{a-b} = \text{the time from noon,}$$

and $$ax = \frac{ac}{a-b} = \text{the distance from } M.$$

1st. If a, b, and c are positive and $a > b$, $\frac{c}{a-b}$ and $\frac{ac}{a-b}$ are both $+$, showing that the couriers will be together after noon and to the east of M. The place is also east of N, since $\frac{ac}{a-b} > c$.

2d. If a, b, and c are positive and $a < b$, $\frac{c}{a-b}$ and $\frac{ac}{a-b}$ are both $-$, showing that the couriers were together before noon and to the west of M.

3d. If a, b, and c are positive and $a = b$, $\frac{c}{a-b}$ and $\frac{ac}{a-b}$ both become ∞, showing that the couriers never have been and never will be together. Their rates being the same, the distance between them is always c.

4th. If $c = 0$ and $a \lesseqgtr b$, $\frac{c}{a-b}$ and $\frac{ac}{a-b}$ are both 0, showing that the couriers are together at noon, M and N now coinciding.

5th. If $c = 0$ and $a = b$, $\frac{c}{a-b}$ and $\frac{ac}{a-b}$ both become $\frac{0}{0}$. As numerators and denominators reduce to 0 by two independent conditions, viz., $c = 0$ and $a = b$, the values of the fractions are indeterminate (Art. 404), and represent any values, showing that the couriers are together all the time.

6th. If a and c are positive and b negative, the time becomes $\frac{c}{a+b}$ and the distance $\frac{ac}{a+b}$. Since these are both $+$, the couriers will be together after noon and to the east of M. The place is between M and N, since $\frac{ac}{a+b} < c$. B's negative rate means that he is traveling to the west.

7th. If b and c are positive and a negative, the time becomes $\frac{c}{-a-b}$ and the distance $\frac{-ac}{-a-b}$, or $\frac{ac}{a+b}$. Since the time is $-$ and the distance $+$, the couriers were together before noon and to the east of M.

8th. If a and c are positive and $b = 0$, the time becomes $\frac{c}{a}$ and the distance c. Hence the couriers will be together after noon and at c miles east of M. The condition $b = 0$ means that B remains at N.

9th. If b and c are positive and $a = 0$, the time becomes $\frac{c}{b}$ and the distance 0. Hence the couriers were together before noon and at M.

If c were made negative, it would locate N west of M.

Thus all the features of the problem are correctly revealed by the formulas for the time and the distance, the analytic results agreeing with the conditions.

13. Divide a into two parts the difference of whose squares shall be d.

What is the character of the parts when $d > a^2$? Why has d neither a maximum nor a minimum value?

14. Two couriers, A and B, are traveling the same road, the former at rate a, and the latter n times as fast. They are at two places c miles apart at the same time. Find the time and the place of their being together.

15. A cistern can be filled by one pipe in m minutes and by another in n minutes, and can be emptied by a third in p minutes. In what time will it be filled if all are left open at once?

16. A and B, traveling the same road, were at two towns m miles apart at the same time. On coming together it was found that A, the faster traveler, had gone n miles, and that their time on the road was equal to the difference of their rates. Find their rates.

17. The loudness of one church bell is m times that of another. If the amount of sound varies directly as the loudness and inversely as the square of the distance, where on the line of the two will the bells be equally well heard, the distance between them being a?

CHAPTER XIX

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS

409. The Differential of a Variable or Function (Arts. 380 and 382) is its rate of change.

If the change is uniform, the rate of change is the same as the actual change during the unit of time; but if the change is not uniform, the rate of change at any instant is the change that *would* take place during the next unit of time *if* it continued uniform at what it is at that instant.

Thus, we say that a body falling from rest has at the end of the first second a velocity of $32\frac{1}{6}$ feet. By this we do not mean that it falls only $32\frac{1}{6}$ feet during the next second, but that if it continued to fall uniformly with the velocity it has at that instant, it *would* fall $32\frac{1}{6}$ feet during the next second. The distance from the falling point is variable and the differential of that variable at the end of the first second is $32\frac{1}{6}$ feet.

410. The differential of a variable is indicated by writing d before it. Thus, the differential of x is written dx, and is read "differential x."

411. If a variable or function is increasing, its differential is positive; if decreasing, negative.

412. Theorem. *Constant terms disappear in differentiating.*

For since a constant admits of no change, the differential of a constant is 0.

413. Theorem. *The differential of the product of a constant and a variable is the constant multiplied by the differential of the variable.*

Dem. Let the function be ax. (1)

Whether x changes uniformly or not, let dx be the amount by which it *would* increase in a unit of time *if*, at any instant, the change should become uniform. Then the state of the function at the end of the unit of time would be

$$a(x + dx) = ax + adx. \tag{2}$$

The difference between the state of the function at the beginning of a unit of time and what, with a uniform change, it *would* be at the end of the unit of time is the differential of the function (Art. 409). Hence, subtracting (1) from (2), we have

$$d(ax) = adx.$$

414. Theorem. *The differential of a polynomial is the algebraic sum of the differentials of its terms.*

DEM. Let the function be $x + y - z$. (1)

Whether x, y, and z change uniformly or not, let dx, dy, and dz be the respective amounts by which they *would* increase in a unit of time *if*, at any instant, the change should become uniform. Then the state of the function at the end of the unit of time would be

$$x + dx + y + dy - (z + dz). \qquad (2)$$

The difference between the state of a function at the beginning of a unit of time and what, with a uniform change, it *would* be at the end of the unit of time is the differential of the function (Art. 409). Hence, subtracting (1) from (2), we have

$$d(x + y - z) = dx + dy - dz.$$

415. Theorem. *The differential of the product of two variables is the sum of the products of each into the differential of the other.*

DEM. Let the function be xy.

Whether x and y change uniformly or not, let dx and dy be the respective amounts by which they *would* increase in a unit of time *if*, at any instant, the changes should become uniform.

Now if x alone were to change, the change in the product would be ydx (Art. 413), and if y alone were to change, the change in the product would be xdy (Art. 413), while if both x and y change, the change in the product at a uniform rate would be the sum of the changes due to these two causes, or $ydx + xdy$. Hence,

$$d(xy) = ydx + xdy.*$$

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* While this is not a rigorous demonstration, it is sufficient to show the truth of the theorem.

416. Theorem. *The differential of the product of several variables is the sum of the products of the differential of each into all the others.*

DEM. Let the function be xyz.

Let $v = xy$. Then $xyz = vz$.

By Art. 415, $$d(vz) = zdv + vdz, \tag{1}$$

and $$dv = ydx + xdy. \tag{2}$$

Substituting in (1) the values of v and dv, we have

$$d(xyz) = z(ydx + xdy) + xydz,$$

$$= yzdx + xzdy + xydz.$$

The same method of reasoning will apply to any number of variables.

417. Theorem. *The differential of a fraction with variable numerator and denominator is the denominator into the differential of the numerator minus the numerator into the differential of the denominator, divided by the square of the denominator.*

DEM. Let the function be $\frac{x}{y}$, and represent this function by u, giving

$$u = \frac{x}{y}.$$

Clearing of fractions,

$$yu = x.$$

Differentiating by Art. 415,

$$ydu + udy = dx,$$

whence $$du = \frac{dx - udy}{y} = \frac{dx - \frac{x}{y}dy}{y} = \frac{ydx - xdy}{y^2}.$$

418. Cor. I. *The differential of a fraction whose numerator is constant is minus the numerator into the differential of the denominator, divided by the square of the denominator.*

For if $x = a$, a constant, then $da = 0$, and $d\left(\frac{a}{y}\right) = -\frac{ady}{y^2}.$

419. Cor. 2. *The differential of a fraction with a constant denominator is the differential of the numerator divided by the denominator.*

For $\frac{x}{b}=\frac{1}{b}x$, and $\frac{1}{b}$ is a constant factor; therefore, by Art. 413,

$$d\left(\frac{x}{b}\right)=d\left(\frac{1}{b}x\right)=\frac{1}{b}dx.$$

420. Theorem. *The differential of a variable having a constant exponent is the product of the exponent, the variable with its exponent diminished by one, and the differential of the variable.*

Dem. Let the function be x^n.

1st. When n is a positive integer.

Now $$x^n = x\cdot x\cdot x\cdots \text{to } n \text{ factors}.$$

Hence, by Art. 416,

$$\begin{aligned} d(x^n) &= (x\cdot x\cdots \text{to } n-1 \text{ factors})\,dx \\ &\quad +(x\cdot x\cdots \text{to } n-1 \text{ factors})\,dx+\cdots \text{to } n \text{ terms}, \\ &= x^{n-1}dx+x^{n-1}dx+\cdots \text{to } n \text{ terms}, \\ &= nx^{n-1}dx. \end{aligned}$$

2d. When n is a positive fraction, as $\frac{p}{q}$.

Let $$y=x^{\frac{p}{q}}.$$

By involution, $$y^q=x^p.$$

By 1st case, $$qy^{q-1}dy=px^{p-1}dx,$$

whence $$dy=\frac{px^{p-1}dx}{qy^{q-1}}=\frac{px^{p-1}dx}{q(x^{\frac{p}{q}})^{q-1}}=\frac{p}{q}x^{\frac{p}{q}-1}dx.$$

3d. When n is negative.

Now $$x^{-n}=\frac{1}{x^n}.$$

Differentiating by Art 418 and 1st case,

$$d(x^{-n})=d\left(\frac{1}{x^n}\right)=\frac{-nx^{n-1}dx}{x^{2n}}=-nx^{-n-1}dx.$$

421. Cor. *The differential of the square root of a variable is the differential of the variable divided by twice the square root of the variable.*

For $$d(\sqrt{x}) = d(x^{\frac{1}{2}}) = \tfrac{1}{2}x^{-\frac{1}{2}}\,dx = \frac{dx}{2\sqrt{x}}.$$

EXAMPLES CIX

Differentiate the following:

1. $y = 2x^3 - 4x^2 + 5x.$

Operation. By Art. 414,

$$dy = d(2x^3) - d(4x^2) + d(5x)$$
$$= 3 \cdot 2x^2dx - 2 \cdot 4xdx + 5dx, \text{ by Arts. 413 and 420.}$$
$$= (6x^2 - 8x + 5)dx.$$

This means that y, or $2x^3 - 4x^2 + 5x$, changes $6x^2 - 8x + 5$ times as fast as x. Hence,

when $x = 0$, y changes 5 times as fast as x;

when $x = 1$, y changes 3 times as fast as x;

when $x = 2$, y changes 13 times as fast as x;

when $x = 3$, y changes 35 times as fast as x.

2. $y = x^3 + 5x^2 - 6x + 7.$

3. $y = x^{-3} - 4x^2 + 25.$

4. $y = 7x^4 + 3x^2 - 6x^{\frac{1}{2}}.$

5. $y = 2x^5 - 5x^{-4} - 5x^2 + 9.$

6. $y = 3(5 + 4x^3)^5.$

Operation. Treating the part in the parenthesis as the variable, we have

$$dy = 5 \cdot 3(5 + 4x^3)^4 3 \cdot 4x^2dx = 180(5 + 4x^3)^4x^2dx.$$

7. $y = 5(2 + 3x^2)^4.$

8. $y = 4(1 - 5x^2)^3.$

9. $y = (a + bx^3)^{\frac{5}{3}}.$

10. $y = 2(3 - x^3)^{-2}.$

11. $u = x^2y^3.$

Operation. By Art. 415,

$$du = y^3d(x^2) + x^2d(y^3) = 2xy^3dx + 3x^2y^2dy.$$

12. $u = x^3y^{\frac{3}{2}}.$

13. $u = x(y^2 - 3).$

14. $u = (x - 1)(y^3 + 2).$

15. $u = x(x + a)^{\frac{5}{2}}.$

16. $y = \dfrac{x^2 - 3}{x^3}.$

OPERATION. By Art. 417,

$$dy = \frac{x^3d(x^2-3)-(x^2-3)d(x^3)}{(x^3)^2}$$

$$= \frac{2x^4dx - 3x^4dx + 9x^2dx}{x^6} = \frac{9-x^2dx}{x^4}.$$

17. $y = \dfrac{1}{(1+x)^3}$.

18. $u = \dfrac{x^3}{y^2}$.

19. $y = \sqrt{3+5x^3}$.

OPERATION. By Art. 421,

$$dy = \frac{d(3+5x^3)}{2\sqrt{3+5x^3}} = \frac{15x^2dx}{2\sqrt{3+5x^3}}.$$

20. $y = \sqrt{1+x}$.

21. $y = 4\sqrt{2x-3x^2}$.

422. **An Equicrescent Variable** is one which changes uniformly, *i.e.*, one which has a constant rate.

423. **A Second Differential** is a differential of a differential, *i.e.*, the rate at which the rate is changing.

The differential of dy is written d^2y, and is read, "second differential y"; while $dx \times dx$, or the square of dx, is written dx^2.

From the function $y = ax^3$,

we have $dy = 3ax^2dx = (3adx)x^2$.

If we regard x as an equicrescent variable, dx is constant; but dy is variable, since it is equal to an expression containing a variable x.

Hence we have

$$d(dy) = d^2y = d[(3adx)x^2] = 3adx \times 2xdx = 6axdx^2.$$

424. **A Third Differential** is the differential of the second differential, and so on.

425. **The First Derivative**, or **the First Differential Coefficient**, is the ratio of the differential of the function to the differential of the variable, and is represented by $\dfrac{dy}{dx}$, or $f'(x)$.

Thus, from $\quad y = x^3 + 3x^2 - 5x,$

we have $\quad dy = (3x^2 + 6x - 5)dx,$

and $\quad \dfrac{dy}{dx} = 3x^2 + 6x - 5,$

in which $3x^2 + 6x - 5$ is the first derivative.

The same thing is expressed by writing

$$f(x) = x^3 + 3x^2 - 5x,$$

and $\quad f'(x) = 3x^2 + 6x - 5.$

426. **The Second Derivative**, or the **Second Differential Coefficient**, is the ratio of the second differential of the function to the square of the differential of the variable, and is represented by $\dfrac{d^2y}{dx^2}$, or $f''(x)$.

427. **The Third Derivative**, or the **Third Differential Coefficient**, is the ratio of the third differential of the function to the cube of the differential of the variable, and is represented by $\dfrac{d^3y}{dx^3}$, or $f'''(x)$, and so on.

EXAMPLES CX

Find the first derivative of each of the following:

1. $y = 2 + 3x - 4x^3.$

OPERATION. $\quad dy = 3\,dx - 12x^2dx,$

whence $\quad \dfrac{dy}{dx} = 3 - 12x^2.$

2. $y = x^3 - 3x^2 + 4.$

3. $y = (x^3 - 5)^3.$

4. $y = \sqrt{x^2 - 2}.$

5. $y = x^2 + x + x^{-3}.$

6. $y = \dfrac{m}{(1+x)^2}.$

7. $y = \dfrac{m}{(x+a)^4}.$

Find the second derivative of each of the following:

8. $y = 5(3 + 4x)^3.$

OPERATION. $\quad f'(x) = 60(3 + 4x)^2,$

and $\quad f''(x) = 480(3 + 4x).$

9. $y = x^2 - 6x + 7.$

10. $y = 3x^4 - 5x^2 + 10.$

11. $y = 3(2 + 5x)^3.$

12. $y = (a + x)^{-3}.$

13. $y = \dfrac{m}{a + x}.$

14. $y = 5(2x^2 - 3)^4.$

Find the first four successive derivatives of each of the following:

15. $y = (a + x)^{-2}$.
16. $y = \dfrac{m}{a + x}$.

17. $y = 3(1 + x)^5$.
18. $y = 3x^4 - 4x^3 + 3x^2 + 6x - 7$.

19. $y = 2(5 - x)^{\frac{1}{3}}$.

20. $y = A + Bx + Cx^2 + Dx^3 + Ex^4$.

428. A Partial Differential of a function of more than one variable is its differential on the hypothesis that all the variables but one become constant.

Thus, from $$u = 3xy + 5x - 4y + 7,$$
we have $$d_x u = 3ydx + 5dx,$$
and $$d_y u = 3xdy - 4dy,$$
the subscripts indicating with reference to which variable we have differentiated.

429. A Partial Derivative, or **a Partial Differential Coefficient,** is the ratio of the partial differential of the function to the differential of that variable with reference to which we have differentiated.

Thus, from the above function we have
$$\frac{du}{dx} = 3y + 5,$$
and $$\frac{du}{dy} = 3x - 4,$$
the denominators indicating with reference to which variable we have differentiated.

430. Theorem. *In an algebraic function of the sum of two variables, x and y, the first, second, third, etc., partial derivatives with reference to x are equal respectively to the first, second, third, etc., partial derivatives with reference to y.**

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* This is true for *any* function of the sum of two variables, but it is proved here only for algebraic functions.

DEM. All algebraic functions of the sum of two variables are included in the form

$$u = m(x + y)^n,$$

in which m and n are positive or negative, integral or fractional.

Now $$d_x u = mn(x + y)^{n-1} dx,$$

and $$d_y u = mn(x + y)^{n-1} dy,$$

which forms differ only in the factors dx and dy; but in forming the derivatives these factors are divided out, giving

$$\frac{du}{dx} = \frac{du}{dy} = mn(x + y)^{n-1}.$$

This is a new function of the sum of two variables, and, as just shown, its first partial derivatives are equal; but these are the second partial derivatives of the original function, giving

$$\frac{d^2u}{dx^2} = \frac{d^2u}{dy^2}.$$

Similarly, $$\frac{d^3u}{dx^3} = \frac{d^3u}{dy^3}, \quad \frac{d^4u}{dx^4} = \frac{d^4u}{dy^4}, \quad \frac{d^nu}{dx^n} = \frac{d^nu}{dy^n}.$$

431. Let the student form the first three successive partial derivatives of $u = 2(x + y)^3$, $u = 6x^4 + 3y^3$, $u = 5(x + y)^{-2}$, $u = \sqrt{x + y}$, $u = 5(x - y)^7$, $u = (x^2 + 3y)^4$, $u = (x + y)^{\frac{2}{3}}$, and observe that the law stated in the theorem holds for all straight functions of the sum of two variables, while it does not for other functions; though in a straight function of the difference of two variables the corresponding partial derivatives are equal except in sign.

CHAPTER XX

DEVELOPMENT OF FUNCTIONS

432. **A Finite Series** is a series (Art. 312) which terminates.

433. **An Infinite Series** is a series which does not terminate, but has an endless succession of terms.

434. An infinite series is **Convergent** when, by taking more and more terms, the sum approaches a finite limit; it is **Divergent** when the sum does not approach a finite limit; and it is **Oscillating**, or **Neutral**, when the sum, though finite, does not approach a determinate limit.

Thus, $$1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$$

is *convergent* when x is numerically less than 1, approaching the limit of $\frac{1}{1-x}$ (Art. 329); *divergent* when x is 1 or greater than 1; *oscillating* when x is -1, the sum being 0 or 1 according as the number of terms is even or odd.

435. NOTE. *Convergent*, when applied to a series, is not synonymous with *decreasing* or *descending* (Art. 313). While every convergent series is decreasing, not every decreasing series is convergent. Thus, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots$ is decreasing, but is not convergent, as shown in Art. 583 (3).

436. **To Develop** or **Expand a Function** is to find a series whose sum is equal to the function. The value of a developed function, therefore, is either the sum of a finite series, or the limit of the sum of an infinite convergent series.

437. There are many ways of developing functions. The student is already familiar with development by division, involution, and evolution.

Thus, by actual division,

$$\frac{1-x^6}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5,$$

a finite series and equal to the function for all values of x; by the binomial theorem (Art. 155),

$$(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5,$$

a finite series and equal to the function for all values of x; and by extracting the square root of $1 - x^2$ we have

$$\sqrt{1-x^2} = 1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \cdots,$$

an infinite series and true only for values of x numerically equal to or less than 1, being divergent for all other values.

438. It often occurs, as in the last example, that a development is true for certain values of the variable, but not for all values.

Thus, by division,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots.$$

Now when x is a proper fraction, either positive or negative, the series is convergent, and, if we treat it as an infinite, decreasing geometrical progression, the formula

$$S = \frac{a}{1-r} \text{ (Art. 329) gives } \frac{1}{1-x},$$

and the series is seen to be equal to the function.

When x is greater than 1, the function (the first member) is a negative fraction, while the series (the second member) is positive and departs farther and farther from the value of the function as more and more terms are taken.

When x is numerically greater than 1, but negative, the function is a positive fraction, while the series is positive or negative according as the number of terms is odd or even, and departs farther and farther from the value of the function as more and more terms are taken.

When $x = -1$, the function is $\frac{1}{2}$, while the series is $1-1+1-1+1-1+\cdots$, which is 0 or 1 according as the number of terms is even or odd.

When $x = 1$, both the function and the series become ∞.

If the order of the terms in the divisor be reversed, we have

$$\frac{1}{-x+1} = -\frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{x^4} - \cdots,$$

and the series equals the function for values of x numerically greater than 1, but not for values numerically less than 1.

439. The subject of *Convergency* is reserved for a subsequent part of the work, and we shall in this chapter concern ourselves only with the *Development*.

SECTION I — INDETERMINATE COEFFICIENTS

440. Indeterminate (or **Undetermined**) **Coefficients** are assumed constant coefficients whose values, not known at the outset, are to be determined in the course of the demonstration of a theorem or the solution of a problem.

441. Theorem of Indeterminate Coefficients. *If two series containing the same variable are equal for all values of the variable that render both series convergent, the coefficients of the same powers of the variable in the two series are equal.*

DEM. Let

$$A + Bx + Cx^2 + Dx^3 + \text{etc.} = A' + B'x + C'x^2 + D'x^3 + \text{etc.}$$

for all values of x which render each series either finite or infinite and convergent; then $A = A'$, $B = B'$, $C = C'$, $D = D'$, etc.

Since the equation is true for all values of x which render each series either finite or infinite and convergent, it is true when $x = 0$. But this gives $A = A'$; and since A and A' are constant, they have the same values, whatever the value assigned to x, so long as each series is either finite or infinite and convergent.

Dropping these equals from the two members and dividing by x,

$$B + Cx + Dx^2 + \text{etc.} = B' + C'x + D'x^2 + \text{etc.}$$

As before, making $x = 0$, we have $B = B'$. Proceeding in the same way, we have $C = C'$, $D = D'$, etc.

Cor. If $A + Bx + Cx^2 + Dx^3 + \text{etc.} = 0$ for all values of x, each of the coefficients A, B, C, etc., is 0.

For we may write

$$A + Bx + Cx^2 + Dx^3 + \text{etc.} = 0 + 0\,x + 0\,x^2 + 0\,x^3 + \text{etc.},$$

whence, by the theorem, $A = 0$, $B = 0$, $C = 0$, etc.

442. The theorem of Art. 441 gives a method, called the Method of Indeterminate (or Undetermined) Coefficients, of developing a function into a series. It consists in assuming a series in ascending powers of the variable with unknown coefficients, and then finding the values of these coefficients by equating those of the same powers of the variable.

Thus, let it be required to develop $\frac{1-3x-x^2}{1-2x-x^2}$.

SOLUTION. Assume

$$\frac{1-3x-x^2}{1-2x-x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\text{etc.}$$

Clearing of fractions,

$$1-3x-x^2=A+\left.\begin{array}{r}B\\-2A\\ \end{array}\right|x+\left.\begin{array}{r}C\\-2B\\-A\end{array}\right|x^2+\left.\begin{array}{r}D\\-2C\\-B\end{array}\right|x^3+\left.\begin{array}{r}E\\-2D\\-C\end{array}\right|x^4+\text{etc.}$$

Equating the coefficients of the same powers of x (Art. 441),

$$\begin{aligned}&A=1;\\&B-2A=-3, \text{ whence } B=-3+2A=-1;\\&C-2B-A=-1, \text{ whence } C=-1+2B+A=-2;\\&D-2C-B=0, \text{ whence } D=2C+B=-5;\\&E-2D-C=0, \text{ whence } E=2D+C=-12.\end{aligned}$$

Substituting these values in the assumed series, we have

$$\frac{1-3x-x^2}{1-2x-x^2}=1-x-2x^2-5x^3-12x^4-\text{etc.}$$

This can readily be verified by actual division.

443. In developing by the Method of Indeterminate Coefficients, the **Law of the Series** becomes evident early in the progress of the work, and then as many more coefficients as desired may be obtained by simply combining, according to the discovered law, the coefficients already found.

Thus, in the example of the last article, we have

$$D=2C+B,$$
$$E=2D+C.$$

It is therefore evident that

$$F=2E+D,$$
$$G=2F+E,$$

and so on, each coefficient after the third being twice the preceding plus the second preceding.

In all cases in developing rational fractions *the law of the series appears in all the equations beyond the one whose second member is the coefficient of the highest power of the variable in the numerator*

of the given fraction; for beyond this the first members all have the same form, and the second members are all 0.

The student should in each example note the law of the series.

EXAMPLES CXI

Expand by the Method of Indeterminate Coefficients the following:

1. $\dfrac{1+2x}{1-x-x^2}$.
2. $\dfrac{2-3x}{1+x+x^2}$.
3. $\dfrac{1+x}{1-x+x^2}$.
4. $\dfrac{1+x}{1+2x+3x^2}$.
5. $\dfrac{5+2x}{1-5x+x^2}$.
6. $\dfrac{1+2x}{1-3x^2}$.
7. $\dfrac{2-3x^2+x^4}{1-x+x^2}$.
8. $\dfrac{1+x+x^2}{1+x^3}$.
9. $\dfrac{1+2x}{2-x-x^2}$.

444. In developing a rational fraction we must assume a series such that, after clearing of fractions, the second member shall contain all the powers of the variable that are found in the first member, which is the numerator of the given fraction Otherwise it would be necessary to equate given finite coefficients with 0, which would be absurd. The exponent with which to begin the assumed series may be determined by noting what actual division would give for the first, or lowest, exponent.

In developing $\dfrac{1-x}{2x^2+3x^3}$ we assume

$$\frac{1-x}{2x^2+3x^3} = Ax^{-2} + Bx^{-1} + C + Dx + Ex^2 + Fx^3 + \text{etc.};$$

or, if we prefer, we may write the expression in the form $\dfrac{1}{x^2}\left(\dfrac{1-x}{2+3x}\right)$, expand the part within the parenthesis by the use of the series beginning with an absolute term, and then multiply each term by $\dfrac{1}{x^2}$.

If in any case we inadvertently assume a wrong series, the fact will appear by the absurd results obtained when the coefficients of the same powers of the variable are equated. If in this example we should assume the series

$$A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.},$$

clear of fractions, and equate the coefficients of the same powers of x, we would have $1 = 0$, $-1 = 0$, etc., which are absurd results.

In expanding such a fraction as $\frac{2x^2 - 3x^3}{1 - x}$, no absurdity would result in assuming the series $A + Bx + Cx^2 + Dx^3 + Ex^4 +$ etc.; but, inasmuch as A and B would in that case be equated with 0, it is shorter to assume the series $Ax^2 + Bx^3 + Cx^4 + Dx^5 +$ etc.; or, if we prefer, we may write the expression in the form $x^2\left(\frac{2 - 3x}{1 - x}\right)$, expand the part within the parenthesis by the use of the usual series, and then multiply each term by x^2.

EXAMPLES CXII

Expand by the Method of Indeterminate Coefficients the following:

1. $\frac{2}{3x^2 - 4x^3}$.
2. $\frac{1 + x - x^2}{x - 2x^2 + 3x^3}$.
3. $\frac{2x}{3 - 2x^2}$.
4. $\frac{1 - 2x^2 - x^3}{x^2 + x^3 - x^4}$
5. $\frac{x - 3x^2 - x^3}{1 - 2x - x^2}$.
6. $\frac{3 - 2x + x^3}{2x^3 - x^4 - 2x^6}$.

445. The Method of Indeterminate Coefficients may also be employed for expanding a radical. If the quantity under the radical sign is a perfect power of the degree of the index, the expansion will give the root. If the quantity is a binomial, the binomial theorem gives a more expeditious method of expansion.

EXAMPLES CXIII

Expand by the Method of Indeterminate Coefficients the following:

1. $\sqrt{1 - 2x + 3x^2}$.

SOLUTION. Assume

$$\sqrt{1 - 2x + 3x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}$$

Squaring both members, we have

$$1 - 2x + 3x^2 = A^2 + 2AB \left| x + B^2 \atop + 2AC \right| x^2 + 2AD \left| x^3 + C^2 \atop {+2AE \atop +2BD} \right| x^4 + \text{etc.}$$

Equating the coefficients of the same powers of x (Art. 441),

$$A^2 = 1, \text{ whence } A = 1 \text{ (using only the plus sign)};$$

$$2\,AB = -2, \text{ whence } B = \frac{-1}{A} = -1;$$

$$2\,AC + B^2 = 3, \text{ whence } C = \frac{3 - B^2}{2\,A} = 1;$$

$$2\,AD + 2\,BC = 0, \text{ whence } D = -\frac{BC}{A} = 1;$$

$$2\,AE + C^2 + 2\,BD = 0, \text{ whence } E = \frac{-C^2 - 2\,BD}{2\,A} = \frac{1}{2}.$$

Substituting these values in the assumed series, we have

$$\sqrt{1 - 2\,x + 3\,x^2} = 1 - x + x^2 + x^3 + \tfrac{1}{2}\,x^4 + \text{etc.}$$

This can readily be verified by extracting in the usual way the square root of $1 - 2\,x + 3\,x^2$.

2. $\sqrt{1 + x + x^2}$. **3.** $\sqrt{1 + x - x^2}$. **4.** $\sqrt{9 + x - 3\,x^2}$.

5. $\sqrt{4 - 4\,x + 13\,x^2 - 6\,x^3 + 9\,x^4}$. **6.** $\sqrt[3]{1 + x + x^2}$.

DECOMPOSITION OF FRACTIONS

446. We have seen (Art. 136) that several fractions may be united into one fraction whose denominator is the lowest common multiple of all the denominators. The converse operation of separating a fraction into partial fractions is sometimes necessary, especially in the operations of the Integral Calculus. The Method of Indeterminate Coefficients furnishes a means of making this separation.

If the numerator of the fraction to be separated is of higher degree than the denominator, it should, by division, be made lower. Thus,

$$\frac{2\,x^3 + 3\,x^2 - 4\,x + 2}{x^2 - 2\,x + 1} = 2\,x + 7 + \frac{8\,x - 5}{x^2 - 2\,x + 1}.$$

Case I

447. *When the denominator is resolvable into equal or unequal factors of the first degree.*

Rule. *Assume the given fraction equal to several fractions with undetermined numerators, and whose denominators are all of the divisors of the denominator of the given fraction.*

Clear the equation of fractions and collect terms.

Equate the coefficients of the same powers of the variable, and determine the values of the assumed numerators.

Substitute these values in the assumed fractions.

Dem. Since the given fraction is the sum of the partial fractions, each denominator must be a divisor of the given denominator; and since any divisor is a *possible* denominator, all the divisors must be used.

The numerator of an assumed fraction must contain only the undetermined constant, since otherwise the assumed fraction would be capable of farther separation; thus, $\frac{Ax}{x-a} = A + \frac{Aa}{x-a}$.

After clearing the equation of fractions, the coefficients of the same powers of the variable are, by Art. 441, equal. The equations thus formed will furnish the values of the assumed numerators.

EXAMPLES CXIV

Decompose the following into partial fractions:

1. $\frac{3x^2 + 7x - 4}{x^3 + x^2 - 2x}$.

Solution. As the divisors of the denominator are x, $x - 1$, and $x + 2$, we assume

$$\frac{3x^2 + 7x - 4}{x^3 + x^2 - 2x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}.$$

Clearing of fractions and collecting terms,

$$3x^2 + 7x - 4 = (A + B + C)x^2 + (A + 2B - C)x - 2A,$$

whence (Art. 441)

$$A + B + C = 3,$$

$$A + 2B - C = 7,$$

$$-2A = -4.$$

From these equations we have

$$A = 2,\ B = 2,\ \text{and}\ C = -1.$$

Substituting these values in the assumed partial fractions, we have

$$\frac{3x^2 + 7x - 4}{x^3 + x^2 - 2x} = \frac{2}{x} + \frac{2}{x-1} - \frac{1}{x+2}.$$

2. $\dfrac{3x^2-7x+6}{(x-1)^3}.$

SOLUTION. As the divisors of the denominator are $(x-1)^3$, $(x-1)^2$, and $x-1$, we assume

$$\frac{3x^2-7x+6}{(x-1)^3}=\frac{A}{(x-1)^3}+\frac{B}{(x-1)^2}+\frac{C}{x-1}.$$

Clearing of fractions and collecting terms,

$$3x^2-7x+6=Cx^2+(B-2C)x+A-B+C,$$

whence (Art. 441)

$$C=3,$$
$$B-2C=-7,$$
$$A-B+C=6.$$

From these equations we have

$$C=3,\ B=-1,\ \text{and}\ A=2.$$

Substituting these values in the assumed partial fractions, we have

$$\frac{3x^2-7x+6}{(x-1)^3}=\frac{2}{(x-1)^3}-\frac{1}{(x-1)^2}+\frac{3}{x-1}.$$

3. $\dfrac{x^2-2}{x-x^3}.$

4. $\dfrac{x+1}{x^2-2x}.$

5. $\dfrac{x+3}{x^2-x-2}.$

6. $\dfrac{x+1}{x^2-7x+12}.$

7. $\dfrac{x^2-11x+26}{(x-3)^3}.$

8. $\dfrac{2x-13}{x^2+10x+25}.$

9. $\dfrac{3x^2-4}{(x+1)^3}.$

10. $\dfrac{5x^2-4x}{(5x-2)^3}.$

11. $\dfrac{13x+10}{6x^3-13x^2-5x}.$

12. $\dfrac{x^2}{x^3+6x^2+11x+6}.$

13. $\dfrac{x^2+3x-8}{x(x+2)^2}.$

14. $\dfrac{3x^3-11x^2+13x-4}{(x^2-x)(x-2)^2}.$

CASE II

448. *When the denominator is resolvable into equal or unequal quadratic factors.*

RULE. *Assume the given fraction equal to the sum of several fractions whose numerators have the form $Mx+N$, and whose de-*

nominators are all the quadratic divisors of the given denominator, and proceed as in Case I.

Dem. The numerator of a partial fraction may contain a term with the first power of the variable as well as an absolute term, since such a fraction, being in the form

$$\frac{Mx+N}{x^2+ax+b} \text{ or } \frac{Mx+N}{(x^2+ax+b)^n},$$

in which, by hypothesis, x^2+ax+b can not be factored, is not capable of farther reduction.

Moreover, if x were not introduced into the numerator, on clearing of fractions the degree of the second member would be lower by two than the denominator of the given fraction, while the first member (the numerator of the given fraction) may be lower by only one. Then in equating coefficients, the coefficient of the highest power of x in the first member, a given constant, would be placed equal to 0, which is absurd. It is for a similar reason that a constant term must be introduced into the numerator of each assumed fraction.

EXAMPLES CXV

Decompose the following into partial fractions:

1. $$\frac{6x^2+5x+4}{x^3+2x^2+2x}.$$

Solution. As the divisors of the denominator are x and x^2+2x+2, we assume

$$\frac{6x^2+5x+4}{x^3+2x^2+2x}=\frac{A}{x}+\frac{Bx+C}{x^2+2x+2}.$$

Clearing of fractions and collecting terms,

$$6x^2+5x+4=(A+B)x^2+(2A+C)x+2A,$$

whence (Art. 441)

$$A+B=6,$$

$$2A+C=5,$$

$$2A=4.$$

From these equations we have $A=2$, $B=4$, and $C=1$.

Substituting these values in the assumed partial fractions, we have

$$\frac{6x^2+5x+4}{x^3+2x^2+2x}=\frac{2}{x}+\frac{4x+1}{x^2+2x+2}.$$

2. $\dfrac{x^3+x-1}{(x^2+2)^2}$.

Solution. As the divisors of the denominator are $(x^2+2)^2$ and x^2+2, we assume

$$\frac{x^3+x-1}{(x^2+2)^2}=\frac{Ax+B}{(x^2+2)^2}+\frac{Cx+D}{x^2+2}.$$

Clearing of fractions and collecting terms,

$x^3+x-1=Cx^3+Dx^2+(A+2C)x+B+2D$, whence (Art. 441),

$$C=1,$$
$$D=0,$$
$$A+2C=1,$$
$$B+2D=-1.$$

From these equations we have $A=-1$, $B=-1$, $C=1$, $D=0$.
Substituting these values in the assumed partial fractions, we have

$$\frac{x^3+x-1}{(x^2+2)^2}=-\frac{x+1}{(x^2+2)^2}+\frac{x}{x^2+2}.$$

3. $\dfrac{42-19x}{x^3-4x^2+x-4}$.

4. $\dfrac{x^2-2x+3}{(x^2+1)^2}$.

5. $\dfrac{1}{x^3-x^2+2x-2}$.

6. $\dfrac{x^3-1}{(x^2-2x+1)(x^2+x+4)}$.

7. $\dfrac{12x^2-x+10}{x^3-1}$.

8. $\dfrac{x^3+2x+2}{(x^2+2)(x^3+x+2)}$.

SECTION II — TAYLOR'S FORMULA

449. Taylor's Formula is a formula for developing a function of the sum of two variables in terms of the ascending powers of one of the variables.

450. Factorial n is the product of all the integral numbers from 1 to n inclusive, and is written $\underline{|n}$.* Thus,

$$\underline{|3}=2\cdot 3,\ \underline{|5}=2\cdot 3\cdot 4\cdot 5,\ \underline{|n}=2\cdot 3\cdot 4\cdots n.$$

451. Prob. *To produce Taylor's Formula.*

Solution. Assume

$$u=f(x+y)=A+By+Cy^2+Dy^3+Ey^4+Fy^5+\text{etc.},$$

* The notation $n!$ is also employed.

in which A, B, C, etc., though independent of y, are dependent on x, since x appears in the development only as involved in these coefficients. These coefficients are, therefore, variable.

Obtaining the successive partial derivatives with reference to y and equating them, according to Art. 430, with the corresponding successive partial derivatives with reference to x, we have

$$\frac{du}{dy} = B + 2\,Cy + 3\,Dy^2 + 4\,Ey^3 + 5\,Fy^4 + \text{etc.} = \frac{du}{dx},$$

$$\frac{d^2u}{dy^2} = 2\,C + 2\cdot 3\,Dy + 3\cdot 4\,Ey^2 + 4\cdot 5\,Fy^3 + \text{etc.} = \frac{d^2u}{dx^2},$$

$$\frac{d^3u}{dy^3} = 2\cdot 3\,D + 2\cdot 3\cdot 4\,Ey + 3\cdot 4\cdot 5\,Fy^2 + \text{etc.} = \frac{d^3u}{dx^3},$$

$$\frac{d^4u}{dy^4} = 2\cdot 3\cdot 4\,E + 2\cdot 3\cdot 4\cdot 5\,Fy + \text{etc.} = \frac{d^4u}{dx^4},$$

$$\text{etc.,} \qquad\qquad \text{etc.,} \qquad\qquad \text{etc.}$$

Since A, B, C, etc., are independent of y, if we obtain their values for one value of y, we shall have them (in form) for all values of y. Representing by u' the value of u when $y = 0$, we have

$$A = u',\ B = \frac{du'}{dx},\ C = \frac{d^2u'}{dx^2}\frac{1}{2},\ D = \frac{d^3u'}{dx^3}\frac{1}{\lfloor 3},\ E = \frac{d^4u'}{dx^4}\frac{1}{\lfloor 4},\ \text{etc.}$$

Substituting these values in the assumed series, we have

$$u = f(x+y) = u' + \frac{du'}{dx}y + \frac{d^2u'}{dx^2}\frac{y^2}{2} + \frac{d^3u'}{dx^3}\frac{y^3}{\lfloor 3} + \frac{d^4u'}{dx^4}\frac{y^4}{\lfloor 4} + \text{etc.},$$

which is Taylor's Formula.

452. Sch. As successive derivatives are represented by f', f'', f''', etc. (Arts. 425, 426, and 427), and as $f(x+y)$ becomes $f(x)$ when $y = 0$, Taylor's Formula may be written

$$u = f(x+y) = f(x) + f'(x)y + f''(x)\frac{y^2}{2} + f'''(x)\frac{y^3}{\lfloor 3} + f^{\text{iv}}(x)\frac{y^4}{\lfloor 4} + \text{etc.}$$

This formula may be stated as a theorem as follows:

Taylor's Formula develops $u = f(x+y)$ into a series in which the 1st term is what the function becomes when $y = 0$; the 2d term is

y times what the 1st *derivative becomes when* $y=0$; *the* 3d *term is* $\frac{y^2}{2}$ *times what the second derivative becomes when* $y=0$; *the* 4th *term is* $\frac{y^3}{\underline{|3}}$ *times what the* 3d *derivative becomes when* $y=0$, *and so on.*

EXAMPLES CXVI

Expand by Taylor's Formula the following:

1. $(x+y)^5$.

SOLUTION. $u=(x+y)^5$, $u'=x^5$, $\frac{du'}{dx}=5x^4$, $\frac{d^2u'}{dx^2}=4\cdot 5x^3$,

$$\frac{d^3u'}{dx^3}=3\cdot 4\cdot 5x^2,\ \frac{d^4u'}{dx^4}=2\cdot 3\cdot 4\cdot 5x,\ \frac{d^5u'}{dx^5}=2\cdot 3\cdot 4\cdot 5.$$

Substituting these values in Taylor's Formula, we have

$$u=(x+y)^5=x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5.$$

2. $(x-y)^7$. **3.** $(x+y)^{\frac{1}{2}}$. **4.** $(x-y)^{-2}$.

5. $(x+y)^{-\frac{2}{3}}$. **6.** $\sqrt[3]{x+y}$. **7.** $\sqrt{(x+y)^3}$.

453. Taylor's Formula is much used for developing a function of a single variable after the variable has taken an increment.

EXAMPLES CXVII

Expand by Taylor's Formula the following, after the variable has taken an increment h.

1. $2x^3-x^2+5x-11$.

SOLUTION. Using the notation of Art. 452,

$$f(x+h)=2(x+h)^3-(x+h)^2+5(x+h)-11,$$

$$f(x)=2x^3-x^2+5x-11,\ f'(x)=6x^2-2x+5,$$

$$f''(x)=12x-2,\ f'''(x)=12.$$

Substituting these values in Taylor's Formula, we have

$$2x^3-x^2+5x-11+(6x^2-2x+5)h+(6x-1)h^2+2h^3.$$

2. $3x^5-2x^2$. **3.** $2x^4-4x^3+x^2-5$.

454. In producing Taylor's Formula we made use of the principle that *in a function of the sum of two variables the corresponding partial derivatives are equal.* The formula will not, therefore,

directly expand such a function as $(3x^2+4y^3)^5$, in which the corresponding partial derivatives, on account of the coefficients and exponents of x and y, are not equal. Indirectly, however, by substituting z for $3x^2$ and v for $4y^3$, the formula will apply; for in that case the coefficients and exponents of x and y will not enter the differentiation. Then, *after expansion*, the values of z and v may be restored.

EXAMPLES CXVIII

Develop by Taylor's Formula the following:

1. $(x^2+3y)^{\frac{1}{2}}$.

SOLUTION. Substituting z for x^2 and v for $3y$, and developing by Taylor's Formula, we have

$$(z+v)^{\frac{1}{2}} = z^{\frac{1}{2}} + \tfrac{1}{2}z^{-\frac{1}{2}}v - \tfrac{1}{8}z^{-\frac{3}{2}}v^2 + \tfrac{1}{16}z^{-\frac{5}{2}}v^3 - \tfrac{5}{128}z^{-\frac{7}{2}}v^4 + \text{etc.}$$

Restoring values,

$$(x^2+3y)^{\frac{1}{2}} = x + \frac{3y}{2x} - \frac{9y^2}{8x^3} + \frac{27y^3}{16x^5} - \frac{405y^4}{128x^7} + \text{etc.}$$

2. $(2x+y^3)^4$. 3. $(x^{-2}+2y)^{-1}$. 4. $(x-2y^2)^{\frac{2}{3}}$.

455. All of the above functions, which are algebraic, can be readily expanded without the use of Taylor's Formula. The formula has its most important application in expanding functions that are not algebraic.

SECTION III — THE BINOMIAL FORMULA

456. The Binomial Formula and applications with positive integral exponents have already been given in Arts. 155 and 159. The proof and applications with any exponent have been reserved for this part of the work.

457. **Prob.** *To produce the Binomial Formula.*

SOLUTION. Applying Taylor's Formula (Art. 451) to the expansion of the form $(x+y)^m$, we have

$$f(x+y) = (x+y)^m, \quad f(x) = x^m,$$
$$f'(x) = mx^{m-1}, \quad f''(x) = m(m-1)x^{m-2},$$
$$f'''(x) = m(m-1)(m-2)x^{m-3},$$
$$f^{\text{iv}}(x) = m(m-1)(m-2)(m-3)x^{m-4}, \text{ etc.}$$

Substituting these values, we have

$$(x+y)^m = x^m + mx^{m-1}y + \frac{m(m-1)}{\lfloor 2}x^{m-2}y^2 + \frac{m(m-1)(m-2)}{\lfloor 3}x^{m-3}y^3$$

$$+ \cdots + \frac{m(m-1)(m-2)\cdots(m-n+2)}{\lfloor n-1}x^{m-n+1}y^{n-1} + \cdots,$$

which is the Binomial Formula.

458. The Binomial Theorem. *In the expansion of a binomial affected with any exponent, the exponent of the leading letter begins in the first term with the exponent of the binomial, and in each succeeding term decreases by* 1; *while the exponent of the second letter begins in the second term with* 1 *and in each succeeding term increases by* 1.

The coefficient of the first term is 1; *that of the second term is the exponent of the binomial; and if the coefficient of any term be multiplied by the exponent of the leading letter in that term and divided by the exponent of the second letter increased by* 1, *the result will be the coefficient of the next term.*

This is deduced by inspection from the Binomial Formula.

EXAMPLES CXIX

Expand by the binomial theorem the following:

1. $(x^2 - 2y)^{\frac{2}{3}}$.

Solution. The theorem (458) applies, not to the exponents and coefficients of x and y in such a function as this, but to x^2 and $-2y$ treated as wholes. To avoid carrying over into following terms factors and signs that do not belong there, it is best to inclose the separate factors in parentheses, using the plus sign between all the terms, and reducing afterward. Thus,

$$(x^2 - 2y)^{\frac{2}{3}} = (x^2)^{\frac{2}{3}} + (\tfrac{2}{3})(x^2)^{-\frac{1}{3}}(-2y) + (-\tfrac{1}{9})(x^2)^{-\frac{4}{3}}(-2y)^2$$

$$+ (\tfrac{4}{81})(x^2)^{-\frac{7}{3}}(-2y)^3 + \text{etc.},$$

$$= x^{\frac{4}{3}} - \tfrac{4}{3}x^{-\frac{2}{3}}y - \tfrac{4}{9}x^{-\frac{8}{3}}y^2 - \tfrac{32}{81}x^{-\frac{14}{3}}y^3 - \text{etc.}$$

2. $(a-b)^6$. 3. $(x-y)^7$. 4. $(1+x)^4$.

5. $(1-y)^5$. 6. $(x+y)^{-2}$. 7. $(x-y)^{-3}$.

8. $(a-x)^{-1}$.

9. $\dfrac{1}{(x+y)^4}$, or $(x+y)^{-4}$.

10. $\dfrac{1}{x+y}$.

11. $(1-a^2)^{\frac{1}{2}}$.

12. $(2+x^3)^{\frac{2}{3}}$.

13. $\left(x+\dfrac{1}{x^2}\right)^6$.

14. $(x^{-2}-y^{\frac{1}{2}})^{-2}$.

15. $(\sqrt{a^3}+4\sqrt[3]{a})^4$.

16. $\sqrt{x+2}$.

17. $\dfrac{1}{(1-x^2)^3}$.

18. $(a^2-x^2)^{\frac{1}{3}}$.

19. $(3a-x^2)^{-3}$.

20. $(a^2-b^2)^{\frac{2}{3}}$.

21. $\dfrac{a}{\sqrt{b^2-c^2x^2}}$.

22. $(1+\sqrt{x})^4+(1-\sqrt{x})^4$.

23. $(1+\sqrt{5})^5+(1-\sqrt{5})^5$.

24. $(1+x-x^2)^4$.

SUG. Write in the form $[(1+x)-x^2]^4$, and regard $1+x$ as one term of a binomial, and $-x^2$ as the other.

25. $(a+b+c)^5$.

26. $(1-x+x^2-x^3)^4$.

CHAPTER XXI

LOGARITHMS

459. The Logarithm of a Number is the exponent by which a fixed number is affected to produce any required number. The fixed number is called the **Base**.

Thus, let 4 be the base; then

$$\begin{aligned}
3 &\text{ is the logarithm of } 64 \text{ to base } 4, \text{ since } 4^3 = 64; \\
2 &\text{ is the logarithm of } 16 \text{ to base } 4, \text{ since } 4^2 = 16; \\
1 &\text{ is the logarithm of } 4 \text{ to base } 4, \text{ since } 4^1 = 4; \\
0 &\text{ is the logarithm of } 1 \text{ to base } 4, \text{ since } 4^0 = 1; \\
-1 &\text{ is the logarithm of } \tfrac{1}{4} \text{ to base } 4, \text{ since } 4^{-1} = \tfrac{1}{4}; \\
-2 &\text{ is the logarithm of } \tfrac{1}{16} \text{ to base } 4, \text{ since } 4^{-2} = \tfrac{1}{16}.
\end{aligned}$$

460. Logarithms are in one system or another according to the base assumed. Only two systems are in common use, viz., the system whose base is 10, called the **Briggean, Briggs,** or **Common System,** and the system whose base is 2.71828 +, called the **Napierian, Natural,** or **Hyperbolic System.** The Common System is used to facilitate numerical calculations, and the Napierian System is much used in abstract mathematical discussions.

When necessary to distinguish different systems, the bases are written as subscripts to the abbreviation log. Thus, $k = \log_a n$ signifies that k is the logarithm of n in the system whose base is a. The student should clearly understand that, by definition, $k = \log_a n$ expresses the same relation between k and n as does $a^k = n$.

461. Theorem. *The logarithm of* 1 *is* 0 *in all systems.*

For, by Art. 70, $a^0 = 1$ for all values of a.

462. Theorem. *In any system whose base is greater than* 1 *the logarithm of* 0 *is* $-\infty$.

For, a being the base, $a^{-\infty} = \dfrac{1}{a^\infty} = 0$. Therefore $\log 0 = -\infty$.

463. Theorem. *Neither* 1 *nor any negative number can be used as the base of a system of logarithms.*

For with 1 as a base we can represent no other number than 1 by its exponents, 1 with *any* exponent being 1.

Again, with a negative base, odd exponents would give only negative numbers, and the corresponding positive numbers would have no logarithms. For example, with -2 as a base, 3 would be the logarithm of -8, since $(-2)^3 = -8$; but $+8$ would have no logarithm, since it cannot be produced by affecting -2 with any exponent.

464. Theorem. *Negative numbers, as such, have no logarithms.*

For a negative number cannot be produced by affecting a positive base with any exponent.

465. Sch. When negative numbers occur in computation by logarithms, they are treated as if they were positive, and the sign of the result is determined by the number of negative factors. It is usual to write a subscript n after the logarithm of a negative number. Thus, 1.425673_n indicates that the number of which 1.425673 is the logarithm is, in itself, negative.

466. The most important use of logarithms is to facilitate the multiplication, division, involution, and evolution of numbers containing several figures. The processes depend on the following principles.

467. Theorem. *The logarithm of the product of two numbers is the sum of their logarithms.*

Dem. Let a be the base of the system, and p and q any two numbers whose logarithms are x and y respectively. Then, by definition,

$$p = a^x \text{ and } q = a^y.$$

Multiplying together the corresponding members of these equations, we have

$$pq = a^x \times a^y = a^{x+y};$$

that is, $$\log pq = x + y = \log p + \log q.$$

468. Cor. In the same way it may be shown that

$$\log pqr \cdots = \log p + \log q + \log r + \cdots.$$

469. Theorem. *The logarithm of the quotient of two numbers is the logarithm of the dividend minus the logarithm of the divisor.*

DEM. Let a be the base of the system, and p and q any two numbers whose logarithms are x and y respectively. Then, by definition,

$$p = a^x \text{ and } q = a^y.$$

Dividing, we have

$$\frac{p}{q} = \frac{a^x}{a^y} = a^{x-y};$$

that is,

$$\log \frac{p}{q} = x - y = \log p - \log q.$$

470. Theorem. *The logarithm of a power of a number is the logarithm of the number multiplied by the index of the power.*

DEM. Let a be the base, and x the logarithm of p.

Then, by definition, $p = a^x.$

Raising both members to any power, as the nth, we have

$$p^n = (a^x)^n = a^{nx};$$

that is,

$$\log p^n = nx = n \log p.$$

471. Theorem. *The logarithm of any root of a number is the logarithm of the number divided by the index of the root.*

DEM. Let a be the base, and x the logarithm of p.

Then, by definition, $p = a^x.$

Extracting the nth root of both members, we have

$$\sqrt[n]{p} = \sqrt[n]{a^x} = a^{\frac{x}{n}};$$

that is,

$$\log \sqrt[n]{p} = \frac{x}{n} = \frac{\log p}{n}.$$

472. It is thus seen that by the use of logarithms the operations of multiplication are replaced by those of addition, division by subtraction, involution by a single multiplication, and evolution by a single division.

473. Only those numbers that are exact powers of the base have integral logarithms; hence the logarithms of most numbers consist of two parts, an integer and a decimal fraction. It is found convenient to write a logarithm so that its fractional part shall be positive, and its integral part positive or negative, as the case may be.

474. **The Characteristic of a Logarithm** is its integral part, and the **Mantissa of a Logarithm** its fractional part, when the logarithm is so written that the fractional part is positive.

475. Prob. *To ascertain the law of the characteristics of logarithms in the common system.*

SOLUTION. Since $10^0 = 1$, $\log 1 = 0$;
since $10^1 = 10$, $\log 10 = 1$;
since $10^2 = 100$, $\log 100 = 2$;
since $10^3 = 1000$, $\log 1000 = 3$;
etc., etc., etc.

Also, since $10^{-1} = \frac{1}{10} = .1$, $\log .1 = -1$;

since $10^{-2} = \frac{1}{10^2} = .01$, $\log .01 = -2$;

since $10^{-3} = \frac{1}{10^3} = .001$, $\log .001 = -3$;

etc., etc., etc.

Thus it is seen, and this is true in any system, that when the logarithms are in arithmetical progression, the numbers are in geometrical progression.

Now the logarithm of any number between 1 and 10 is between 0 and 1, *i.e.*, $0 +$ a fraction; the logarithm of any number between 10 and 100 is between 1 and 2, *i.e.*, $1 +$ a fraction; the logarithm of any number between 100 and 1000 is between 2 and 3, *i.e.*, $2 +$ a fraction; and so on. Hence *the characteristic of the logarithm of a number > 1 is one less than the number of figures in the integral part of the number.*

Again, the logarithm of any number between 1 and .1 is between 0 and -1, *i.e.*, $-1 +$ a fraction; the logarithm of any

number between .1 and .01 is between -1 and -2, *i.e.*, $-2+$ a fraction; the logarithm of any number between .01 and .001 is between -2 and -3, *i.e.*, $-3+$ a fraction; and so on. Hence *the characteristic of a logarithm of a decimal fraction is negative, and numerically* 1 *greater than the number of* 0's *before the first significant figure.*

476. Characteristics are not written in tables of logarithms, but are to be supplied according to the above principles.

477. When the characteristic of a logarithm is negative, the minus sign is written, not before it, but over it, since the characteristic alone is negative. Thus $\log .00167 = \bar{3}.222716$.

478. Theorem. *Whatever change be made in the position of the decimal point in a number, the mantissa of its logarithm in the common system remains the same.*

Dem. Changing the position of the decimal point is equivalent to multiplying or dividing by some power of 10. Now

$$\log (k \times 10^n) = \log k + \log 10^n = \log k + n \log 10 = \log k + n,$$

$$\log (k \div 10^n) = \log k - \log 10^n = \log k - n \log 10 = \log k - n.$$

Hence the logarithms of k, $k \times 10^n$, and $k \div 10^n$ differ only by an integer n; *i.e.*, the mantissæ of the logarithms of these numbers, which differ only in the position of the decimal point, are the same.

479. Sch. The characteristic, in the common system, *characterizes* the number in showing between what two consecutive powers of 10 it lies, and depends, therefore, simply on the position of the decimal point; while the mantissa, which means *an addition*, is the part added to the characteristic to give the approximate logarithm, and depends simply on the sequence of figures.

480. Theorem. *The differential of the logarithm of a variable is the differential of the variable multiplied by a constant, called the modulus of the system, divided by the variable.*

In the Napierian system, the modulus being 1, *the differential of the logarithm of a variable is the differential of the variable divided by the variable.*

Dem. Let $$y = x^n, \tag{1}$$

n being any arbitrary constant. Then (Art. 420),

$$dy = nx^{n-1}\,dx = n\frac{y}{x}\,dx,$$

or $$\frac{dy}{y} = n\,\frac{dx}{x}. \tag{2}$$

From (1) we have, by Art. 470,

$$\log y = n \log x.$$

Hence from Art. 413, whatever the differentials of $\log y$ and $\log x$,

$$d(\log y) = nd(\log x). \tag{3}$$

Dividing (3) by (2) to eliminate n,

$$\frac{d(\log y)}{\dfrac{dy}{y}} = \frac{d(\log x)}{\dfrac{dx}{x}}.$$

Hence the ratio of $d(\log y)$ to $\dfrac{dy}{y}$ is the same as that of $d(\log x)$ to $\dfrac{dx}{x}$, whatever the values of x and y.

Represent this constant ratio by m. Then

$$d(\log y) = \frac{mdy}{y}, \text{ and } d(\log x) = \frac{mdx}{x}.$$

481. **The Logarithmic Series** is the development of $\log(1 + x)$ in ascending powers of x. It reveals several important properties of logarithms, and, when rendered convergent, furnishes a means of computing the logarithms of numbers.

482. **Prob.** *To produce the logarithmic series.*

Solution. Let $u = \log(z + x)$; then (Art. 451)

$$u' = \log z,\ \frac{du'}{dz} = \frac{m}{z},\ \frac{d^2u'}{dz^2} = -\frac{m}{z^2},$$

$$\frac{d^3u'}{dz^3} = \frac{2\,m}{z^3},\ \frac{d^4u'}{dz^4} = -\frac{2 \cdot 3\,m}{z^4},\ \text{etc.}$$

Substituting these values in Taylor's Formula (Art. 451), we have

$$u = \log(z + x) = \log z + \frac{m}{z}x - \frac{m}{z^2}\frac{x^2}{2} + \frac{m}{z^3}\frac{x^3}{3} - \frac{m}{z^4}\frac{x^4}{4} + \text{etc.}$$

Making $z = 1$, this becomes

$$\log(1 + x) = m\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{etc.}\right),$$

which is the logarithmic series.

483. Sch. 1. In the above series, if we let x remain the same and express the logarithms of $1 + x$ in two different systems whose bases are a and b, $\log_a(1 + x)$ is not equal to $\log_b(1 + x)$. But the series in the parenthesis is the same in the two cases; hence m is different in the two cases, *i.e.*, m *depends on the base and characterizes the system.* This factor, which is constant in the same system but different in different systems, has been named the *modulus.*

484. Sch. 2. It is evident that, in establishing a system of logarithms, we may either select the modulus and let the base be determined by the mutual relation, or select the base and let the modulus be determined by the mutual relation. In the Napierian system (so called) 1 was selected as the modulus, and this gave for the base 2.718281828+; while in the Briggs system 10 was selected as the base, and this gave for the modulus .43429448+.

485. Theorem. *The logarithms of the same number in different systems are to each other as the moduli of those systems.*

Dem. Letting a and a' be the bases, and m and m' the moduli of two systems, we have, from Art. 482,

$$\log_a(1 + x) = m\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{etc.}\right), \qquad (1)$$

$$\log_{a'}(1 + x) = m'\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{etc.}\right). \qquad (2)$$

Dividing (1) by (2),

$$\frac{\log_a(1 + x)}{\log_a(1 + x)} = \frac{m}{m'}. \qquad (3)$$

486. Cor. *The logarithm of a number in the Napierian system multiplied by the modulus of any other system will give the logarithm of the same number in the latter system.*

487. Prob. *To render the Napierian logarithmic series convergent.*

SOLUTION. As the logarithmic series (Art. 482) is divergent for $x > 1$, it cannot, until transformed into a convergent series, be used for computing the logarithms of numbers.

For the Napierian system, whose base is usually represented by e, the series becomes, when x is positive,

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{etc.,} \tag{1}$$

and when x is negative,

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{4} - \text{etc.} \tag{2}$$

Subtracting (2) from (1), we have

$$\log_e(1+x) - \log_e(1-x) = \log_e\frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \text{etc.}\right).$$

Letting $$\frac{1+x}{1-x} = \frac{p}{q},$$

which gives by solving the equation

$$x = \frac{p-q}{p+q},$$

and substituting,

$$\log_e\frac{p}{q} = \log_e p - \log_e q$$

$$= 2\left[\frac{p-q}{p+q} + \frac{1}{3}\left(\frac{p-q}{p+q}\right)^3 + \frac{1}{5}\left(\frac{p-q}{p+q}\right)^5 + \text{etc.}\right],$$

or

$$\log_e p = \log_e q + 2\left[\frac{p-q}{p+q} + \frac{1}{3}\left(\frac{p-q}{p+q}\right)^3 + \frac{1}{5}\left(\frac{p-q}{p+q}\right)^5 + \text{etc.}\right], \tag{A}$$

which is seen to be a convergent series.

488. Prob. *To compute the logarithms of* 1, 2, 3, 4, 5, *etc.*

SOLUTION. It is necessary to compute the logarithms of prime numbers only, since the logarithm of a composite number is equal to the sum of the logarithms of its factors (Art. 467).

The logarithm of 1 is 0 (Art. 461).

To find the Napierian log 2, in (A) make $p=2$ and $q=1$.

To find the Napierian log 3, make $p=3$ and $q=2$, and so on. We thus have

$$\log_e 2 = \log_e 1 + 2\left(\frac{1}{3}+\frac{1}{3\cdot 3^3}+\frac{1}{5\cdot 3^5}+\frac{1}{7\cdot 3^7}+\frac{1}{9\cdot 3^9}+\text{etc.}\right)$$
$$0.69314718,$$

$$\log_e 3 = \log_e 2 + 2\left(\frac{1}{5}+\frac{1}{3\cdot 5^3}+\frac{1}{5\cdot 5^5}+\frac{1}{7\cdot 5^7}+\text{etc.}\right) = 1.09861228,$$

$$\log_e 4 = 2\log_e 2 = 1.38629436,$$

$$\log_e 5 = \log_e 4 + 2\left(\frac{1}{9}+\frac{1}{3\cdot 9^3}+\frac{1}{5\cdot 9^5}+\frac{1}{7\cdot 9^7}+\text{etc.}\right) = 1.60943790,$$

and so on.

The common logarithms of the same numbers may now be found by multiplying the Napierian logarithms by the modulus of the common system (Art. 486).

489. Tables of such logarithms have been prepared with great care and are easily obtainable by the student and the practical computer. Bremiker's for six places and Vega's for seven are standard tables. For some refined astronomical calculations ten-place logarithms are necessary, while for many purposes four-place and five-place logarithms are sufficient.

490. Theorem. *The modulus of any system is the reciprocal of the Napierian logarithm of the base of that system, and in the common system is* .43429448; *and the Napierian base is the number of which the modulus of any system is the logarithm in that system, and is* 2.71821828.

DEM. In equation (3) of Art. 485, letting $1+x=a=$ base, we have

$$\frac{\log_a a}{\log_e a}=\frac{m}{1}, \text{ or } m=\frac{1}{\log_e a}.$$

In the common system this becomes

$$m=\frac{1}{\log_e 10}=\frac{1}{\log_e 2+\log_e 5}=\frac{1}{2.30258508}=.43429448.$$

Again, by the same equation,

$$\frac{\log_{10} e}{\log_e e} = \frac{m}{1},$$

or $$\log_{10} e = m \log_e e = m = .43429448.$$

Finding from a table of common logarithms the number of which .43429448 is the logarithm, we have

$$e = 2.71821828.$$

491. An Exponential Equation is an equation in which the unknown quantity occurs as an exponent.

492. Prob. *To solve an exponential equation.*

SOLUTION. By definition the equation has the form

$$a^x = b.$$

Taking the logarithms of both members, we have

$$x \log a = \log b,$$

whence $$x = \frac{\log b}{\log a}.$$

493. Prob. *The principal p, rate r, and time t being given, to find the amount a, at compound interest.*

SOLUTION. We have

for 1 year, $a = p + pr = p(1 + r)$,

for 2 years, $a = p(1 + r) + p(1 + r)r = p(1 + r)^2$,

for 3 years, $a = p(1 + r)^2 + p(1 + r)^2 r = p(1 + r)^3$,

and so on. Hence, for t years,

$$a = p(1 + r)^t. \qquad (1)$$

Taking the logarithms of both members, we have

$$\log a = \log p + t \log(1 + r). \qquad (2)$$

494. Cor. I. *If the interest is compounded m times annually, formulas* (1) *and* (2) *become*

$$a = p\left(1 + \frac{r}{m}\right)^{mt},$$

$$\log a = \log p + mt \log\left(1 + \frac{r}{m}\right).$$

Cor. 2. *From* (2) *we deduce*

$$\log p = \log a - t \log (1 + r),$$

$$\log (1 + r) = \frac{\log a - \log p}{t},$$

$$\log t = \frac{\log a - \log p}{\log (1 + r)}.$$

495. Prob. *To find the present worth, S, of an annuity, a, running t years.*

Solution. Money being worth r per cent,

the present worth of 1st payment $= \dfrac{a}{1 + r}$,

the present worth of 2d payment $= \dfrac{a}{(1 + r)^2}$,

the present worth of 3d payment $= \dfrac{a}{(1 + r)^3}$,

and so on. Hence the present worth of the whole is the sum of a geometrical progression in which the first term is $\dfrac{a}{1 + r}$, the ratio is $\dfrac{1}{1 + r}$, and the number of terms is t. Substituting in the formula of Art. 327 and reducing, we have

$$S = \frac{a[(1 + r)^t - 1]}{r(1 + r)^t},$$

whence $\log S = \log a + \log [(1 + r)^t - 1] - \log r - t \log (1 + r)$.

EXAMPLES CXX

Express the following operations by logarithms:

1. $y = \sqrt{\dfrac{a^2 - x^2}{1 + x}}$.

Solution. $y = \sqrt{\dfrac{a^2 - x^2}{1 + x}} = \left(\dfrac{(a + x)(a - x)}{1 + x}\right)^{\frac{1}{2}}$.

Hence, $\log y = \frac{1}{2} [\log (a + x) + \log (a - x) - \log (1 + x)]$.

2. $y = x^{\frac{2}{3}}(1 - x^2)^{\frac{1}{2}}$.

3. $y = \sqrt{\dfrac{s(s - a)}{bc}}$.

4. $y = \sqrt[3]{\dfrac{ab^2c^4}{d^5}}$.

5. $y = \sqrt{\dfrac{a^6 - a^4b^2}{1 + b}}$.

6. $y = \dfrac{\sqrt[3]{x - x^2}}{\sqrt{z}}$.

7. $y = \sqrt{\dfrac{a^m b^p}{c^t}}$.

8. $y = \dfrac{\sqrt[3]{a}\sqrt{a^2 - b^2}}{\sqrt{a}\sqrt[3]{a^2 - b^2}}$.

9. $\sqrt[3]{\dfrac{x^3 - 6x^2 + 8x}{x^2 - x - 2}}$.

Differentiate the following:

10. $y = \log\sqrt{1 + x}$.

Solution. $y = \log\sqrt{1 + x} = \log(1 + x)^{\frac{1}{2}} = \frac{1}{2}\log(1 + x)$.

Hence, by Arts. 413 and 480,

$$dy = \frac{mdx}{2(1 + x)}.$$

Or, differentiating without first changing the form, we have (Arts. 421 and 480),

$$dy = \frac{\dfrac{mdx}{2\sqrt{1 + x}}}{\sqrt{1 + x}} = \frac{mdx}{2(1 + x)}.$$

11. $y = \log ax$.

12. $y = \log(1 - x)$.

13. $y = \log x^3$.

14. $y = \log\dfrac{a}{x}$.

15. $y = \log(a^2 - x^2)$.

16. $y = \log(1 + x^2)^2$.

Solve the following equations:

17. $a^{bx} = c$.

18. $a^x = \dfrac{1}{b}$.

19. $ba^{cx} = d$.

20. $10^{2x} - 6 \cdot 10^x = 7$.

21. $\begin{cases} 10^{x+y} = 1000, \\ 10^{x-y} = 100. \end{cases}$

22. $\begin{cases} a^{x+y} = m, \\ b^{x-y} = n. \end{cases}$

496. Following are two sample pages from a table of logarithms. The characteristics are not written in the table, but are to be supplied by the principles of Art. 475.

497. *To find from the table the mantissa of the logarithm of a number.*

In all cases the decimal point is disregarded, as it has to do only with the characteristic (Art. 478). The logarithms of the

Logarithms of Numbers

N	0	1	2	3	4	5	6	7	8	9	Diff.	
770	886491	6547	6604	6660	6716	6773	6829	6885	6942	6998		**57**
771	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561	**1**	6
772	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	**2**	11
773	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	**3**	17
774	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	**4**	23
775	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806	**5**	29
776	9862	9918	9974	**30	**86	*141	*197	*253	*309	*365	**6**	34
777	890421	0477	0533	0589	0645	0700	0756	0812	0868	0924	**7**	40
778	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482	**8**	46
779	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	**9**	51
780	892095	2150	2206	2262	2317	2373	2429	2484	2540	2595		**56**
781	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151	**1**	6
782	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706	**2**	11
783	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	**3**	17
784	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	**4**	22
785	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367	**5**	28
786	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	**6**	34
787	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	**7**	39
788	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	**8**	45
789	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	**9**	50
790	897627	7682	7737	7792	7847	7902	7957	8012	8067	8122		**55**
791	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	**1**	6
792	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	**2**	11
793	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766	**3**	17
794	9821	9875	9930	9985	**39	**94	*149	*203	*258	*312	**4**	22
795	900367	0422	0476	0531	0586	0640	0695	0749	0804	0859	**5**	28
796	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	**6**	33
797	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	**7**	39
798	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	**8**	44
799	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	**9**	50
800	903090	3144	3199	3253	3307	3361	3416	3470	3524	3578		**54**
801	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120	**1**	5
802	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	**2**	11
803	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	**3**	16
804	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	**4**	22
805	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281	**5**	27
806	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	**6**	32
807	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	**7**	38
808	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	**8**	43
809	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	**9**	49
810	908485	8539	8592	8646	8699	8753	8807	8860	8914	8967		**53**
811	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	**1**	5
812	9556	9610	9663	9716	9770	9823	9877	9930	9984	**37	**2**	11
813	910091	0144	0197	0251	0304	0358	0411	0464	0518	0571	**3**	16
814	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104	**4**	21
815	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637	**5**	27
816	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	**6**	32
817	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	**7**	37
818	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	**8**	42
819	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761	**9**	48
N	0	1	2	3	4	5	6	7	8	9	Diff.	

Logarithms of Numbers

N	0	1	2	3	4	5	6	7	8	9	Diff.	
820	913814	3867	3920	3973	4026	4079	4132	4184	4237	4290		**53**
821	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819	**1**	5
822	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	**2**	11
823	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	**3**	16
824	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	**4**	21
825	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927	**5**	27
826	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	**6**	32
827	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	**7**	37
828	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	**8**	42
829	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	**9**	48
830	919078	9130	9183	9235	9287	9340	9392	9444	9496	9549		**52**
831	9601	9653	9706	9758	9810	9862	9914	9967	**19	**71	**1**	5
832	920123	0176	0228	0280	0332	0384	0436	0489	0541	0593	**2**	10
833	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114	**3**	16
834	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	**4**	21
835	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154	**5**	26
836	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674	**6**	31
837	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	**7**	36
838	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710	**8**	42
839	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	**9**	47
840	924279	4331	4383	4434	4486	4538	4589	4641	4693	4744		**51**
841	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261	**1**	5
842	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776	**2**	10
843	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	**3**	15
844	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	**4**	20
845	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319	**5**	26
846	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	**6**	31
847	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	**7**	36
848	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	**8**	41
849	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	**9**	46
850	929419	9470	9521	9572	9623	9674	9725	9776	9827	9879		**50**
851	9930	9981	**32	**83	*134	*185	*236	*287	*338	*389	**1**	5
852	930440	0491	0542	0592	0643	0694	0745	0796	0847	0898	**2**	10
853	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	**3**	15
854	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	**4**	20
855	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423	**5**	25
856	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	**6**	30
857	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	**7**	35
858	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943	**8**	40
859	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	**9**	45
860	934498	4549	4599	4650	4700	4751	4801	4852	4902	4953		**49**
861	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457	**1**	5
862	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960	**2**	10
863	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463	**3**	15
864	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966	**4**	20
865	7016	7066	7117	7167	7217	7267	7317	7367	7418	7468	**5**	25
866	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	**6**	29
867	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470	**7**	34
868	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970	**8**	39
869	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	**9**	44
N	0	1	2	3	4	5	6	7	8	9	Diff.	

numbers 78, 780, 7800, 7.8, .78, .078, .0078, all have the same mantissa.

In all cases the first three figures of the number are found in the column headed **N**, and the mantissa of the logarithm is found in the same horizontal line with these.

If the number has but three figures, the mantissa is in the column headed **O**. Thus log 793 = 2.899273.

If the number has four figures, the last four figures of the mantissa are in the column headed with the fourth figure of the number, and the two initial figures are in the column headed **O**. The initial figures are never taken from a lower line unless asterisks occupy one or more places of the last four figures of the mantissa, in which case the places of the asterisks are to be supplied with 0's. Thus,

$$\log 8426 = 3.925621, \text{ and } \log 7945 = 3.900094.$$

If the number has more than four figures, the mantissa of the logarithm for the first four figures is found as just explained, and additions are made to it for the other figures by means of one of the auxiliary tables in the margin, the one to be used being headed with the difference between the mantissa just found and the one next larger in the table. The correction for the 5th figure of the number is as given in the auxiliary table; the correction for the 6th figure is one tenth of that given in the same table, and so on. Thus, to find the logarithm of 8576479, we proceed as follows:

Mantissa of log 8576	=	.933285
Correction for 4 in 5th place	=	20
Correction for 7 in 6th place	=	35
Correction for 9 in 7th place	=	45
Log 8576479	=	6.933309

The corrections are found in the auxiliary table headed 50, this being the difference (in the last two decimal places) between the mantissa taken and the next larger.

EXAMPLES CXXI

Find from the table the logarithms of the following numbers:

1. 798.	**2.** 8.62.	**3.** 7784.
4. 82.36.	**5.** .8127.	**6.** 8516.
7 78754.	**8.** 84.936.	**9.** .077284.
10. 8682.56.	**11.** 78.5716.	**12.** .00859465.

498. *To find from the table a number whose logarithm is given.*

We find in the table the next lower mantissa. The first three figures of the number will be in the same horizontal line in the column headed N, and the fourth figure at the top of the column in which the last four figures of the next lower mantissa are found. The remaining figures are found by one of the auxiliary tables. Thus, to find x when $\log x = 2.904567$, we proceed as follows:

The next lower mantissa is .904553, which corresponds to the number 8027. The difference between this mantissa and the next higher is 54 (in the last two decimal places), and this shows which of the auxiliary tables to use. The difference between $\log x$ and log 8027 is 14. This exact difference is not found in the auxiliary table headed 54, but the next lower difference, 11, gives 2 for the 5th figure of the number. This leaves a difference of $14 - 11 = 3$ still to be provided for. As the differences for the 6th figure are only one tenth of the differences in the auxiliary table, we multiply our difference by 10 (this is the same as dividing the differences in the auxiliary table by 10), giving 30. The next smaller difference in the table is 27, which gives 5 for the 6th figure of the number. The characteristic 2 shows that the number has three figures to the left of the decimal point. Hence $x = 802.725$.

EXAMPLES CXXII

Find from the table the numbers of which the following are the logarithms:

1. 3.919758.	**2.** 2.899137.
3. 0.912451.	**4.** 4.938343.

5. 1.902140. **6.** $\bar{1}.922995$.

7. $\bar{2}.888596$. **8.** $\bar{3}.930272$.

9. 5.934953. **10.** 5.913814.

11. $\bar{3}.903090$. **12.** 5.913578.

Solve the following by logarithms:

13. $7746 : 8334 :: 8027 : x$.

14. $77.53 : 821.6 :: 8.097 : x$.

15. $\dfrac{80.75 \times 82.79 \times 791.7}{8.692 \times 770.4}$.

16. $\sqrt[10]{791673000}$. **17.** $8^x = 79846$.

18. $7.94^x = 86978$. **19.** $8.6^x = 7896.24$.

Solution of 17th. Passing to logarithms, we have

$$x \log 8 = \log 79846,$$

whence $$x = \frac{\log 79846}{\log 8} = \frac{4.902253}{.90309} = 5.4283.$$

CHAPTER XXII

INDETERMINATE EQUATIONS OF THE FIRST DEGREE

499. An Indeterminate Equation is an equation having two or more unknown quantities, this equation expressing the only condition imposed upon the unknown quantities. Hence,

A Set of Equations is Indeterminate when it contains more unknown quantities than equations; for by elimination the set may be reduced to a single equation containing two or more unknown quantities.

As shown in Art. 380, the unknown quantities of such an equation are variables and admit of an infinite number of values.

Thus, in the equation $$5x - 3y = 11,$$
we may give to one of the variables any value we please and find for the other such a value as will satisfy the equation.

Again, from the equations
$$3x - y + 4z = 22,$$
$$4x + 3y - 2z = 19,$$
we have by eliminating z, $$11x + 5y = 60,$$
in which x and y admit of an infinite number of values.

500. A single equation having more than one unknown quantity is indeterminate in a less general sense, *if an additional condition not capable of being expressed in an equation is imposed.*

Thus, let it be required to find the *positive integral* values of x and y in
$$7x + 5y = 118.$$
The introduction of the condition that the values are to be *positive integers* greatly restricts the number. In this equation
$$\text{when } x = 4, \quad y = 18,$$
$$\text{when } x = 9, \quad y = 11,$$
$$\text{when } x = 14, \; y = 4,$$

and it may be shown that these are the only positive integers that will satisfy the equation.

The discussion here will be confined to positive integral values of the unknown quantities.

501. Any equation of the first degree containing two unknown quantities can be reduced to one of the forms $ax \pm by = \pm c$, in which a, b, c are positive integers. The form $ax + by = -c$ is not satisfied for any positive integral values of x and y, and the form $ax - by = -c$ is equivalent to $by - ax = c$; hence we need to consider only the forms $ax \pm by = c$.

502. Theorem. *The forms $ax \pm by = c$ have no positive integral values of x and y, if a and b have a common factor not contained in c.*

DEM. Dividing both members by this common factor, the second member is fractional for all values of x and y, while the first member is integral for all positive integral values of x and y; hence the equations are not satisfied for such values.

503. Theorem. *The number of positive integral values of x and y in the form $ax-by=c$ (a and b being prime to each other) is limited.*

DEM. Solving for x, we have

$$x = \frac{c - by}{a}.$$

Now we can use only those positive integral values of y that will render $c - by$ positive (*i.e.*, $by \not> c$) and divisible by a; hence the number of positive integral values of both x and y is limited.

504. Theorem. *The number of positive integral values of x and y in the form $ax - by = c$ (a and b being prime to each other) is infinite.*

DEM. Solving for x, we have

$$x = \frac{c + by}{a}.$$

No positive integral value of y will render x negative, and an infinite number of such values will render $c + by$ divisible by a; hence the number of positive integral values of both x and y is infinite.

EXAMPLES CXXIII

Find the positive integral values of x and y in each of the following:

1. $5x + 9y = 37.$

SOLUTION. Solving for x in terms of y, we have

$$x = \frac{37 - 9y}{5}. \qquad (1)$$

Since x must be positive,

$$9y \not> 37,$$

whence

$$y \not> 4\tfrac{1}{9},$$

and as y must be integral, it cannot be greater than 4.

Now (1) may be written in the form

$$x = 7 - y + \frac{2 - 4y}{5} = 7 - y + 2\frac{1 - 2y}{5}. \qquad (2)$$

To make x integral $\frac{1 - 2y}{5}$ must be integral. The only value of y not greater than 4 that will make $\frac{1 - 2y}{5}$ integral is 3. This substituted in either (1) or (2) gives $x = 2$.

2. $8x + 13y = 138.$

SOLUTION. Solving for x in terms of y, we have

$$x = \frac{138 - 13y}{8}. \qquad (1)$$

Since x must be positive,

$$13y \not> 138,$$

whence

$$y \not> 10.$$

Now (1) may be written in the form

$$x = 17 - y + \frac{2 - 5y}{8}. \qquad (2)$$

When it is not easy to see by inspection all the values of y that will render the fractional part integral, we may proceed as follows:

Since $\frac{2 - 5y}{8}$ must be an integer, any integral number of times this quantity will be an integer. Let us multiply by the smallest number that will give a remainder 1 (or -1) on dividing the coefficient of y by the denominator. This multiplier in this case is 5, and we have

$$\frac{10 - 25y}{8} = 1 - 3y + \frac{2 - y}{8}. \qquad (3)$$

It is now easy to see that 2 and 10 are the only two numbers not greater than 10 that will make $\frac{2-y}{8}$ integral, and the corresponding values of x are 14 and 1.

Or, if we choose, we may represent $\frac{2-y}{8}$ by the undetermined integer m, giving

$$\frac{2-y}{8}=m,$$

whence $$y=2-8m, \tag{4}$$

and from (1) $$x=14+13m. \tag{5}$$

It is seen that x and y are integral for all integral values of m, and (4) and (5) constitute what is called *the general solution in integers* of the given equation ; but (4) shows that only 0 and negative values of m will give positive values for y, and (5) shows that m cannot be smaller than -1. Hence 0 and -1 are the only admissible values of m.

Now when $m=0$, $x=14$, and $y=2$;

and when $m=-1$, $x=1$, and $y=10$.

3. $13x-30y=61$.

Solution. From Art. 504 we know that x and y have an infinite number of values. Solving for x in terms of y, we have

$$x=\frac{61+30y}{13}=4+2y+\frac{9+4y}{13}.$$

Multiplying the numerator of the last fraction by 10 and reducing to a mixed quantity, we have

$$\frac{90+40y}{13}=6+3y+\frac{12+y}{13}.$$

The values of y that make $\frac{12+y}{13}$ integral are 1, 14, 25, etc., and the corresponding values of x are 7, 37, 67, etc.

4. $14x-21y=32$.

As proved in Art. 502, the solution is impossible.

5. $3x+7y=10$.

6. $3x+5y=26$.

7. $5x-14y=11$.

8. $2x+4y=10$.

9. $27x+18y=47$.

10. $4x-19y=23$.

11. $3x+7y=58$.

12. $13x+2y=119$.

13. $14x-5y=7$.

14. $28x-35y=23$.

15. $\begin{cases}2x+3y-5z=-8,\\5x-y+4z=21.\end{cases}$

16. $\begin{cases}20x-21y=38,\\3y+4z=34.\end{cases}$

PROBLEMS LEADING TO INDETERMINATE EQUATIONS

EXAMPLES CXXIV

1. A man employed two squads of men, paying \$ 3 a day to each of the first squad and \$ 4 a day to each of the second. He paid them all \$ 56 a day. How many belonged to each squad?

2. A dairyman paid \$ 752 for cows, \$ 37 each for Jerseys and \$ 23 each for Durhams. How many of each did he buy?

3. In how many ways can a debt of \$ 43 be paid with 2-dollar and 5-dollar bills?

4. A man invested \$ 10,000 in town lots, paying \$ 190 each for those in one locality, and \$ 130 each for those in another. Find the number in each locality.

5. In how many ways can a debt of £ 50 be discharged with guineas and 3-shilling pieces? £ 51?

6. What fraction becomes $\frac{2}{3}$ when its numerator is doubled and its denominator is increased by 7?

7. How many 4-pound and 6-pound weights must be used to weigh 45 pounds?

8. Divide 136 into two parts, one of which when divided by 5 leaves a remainder 2, and the other divided by 8 leaves a remainder 3.

9. How many times each must a 7-inch stick and a 13-inch stick be applied to measure 4 feet?

10. A owes B a shilling. A has only sovereigns, and B has only dollars, worth 4*s.* 3*d.* each. How can A most easily pay B?

11. A wholesale grocer received \$ 180 for 15 barrels of sirup, the prices being \$ 10, \$ 11, and \$ 13 per barrel. How many of each kind did he sell?

12. A farmer paid \$ 160 for pigs, sheep, and calves. The pigs cost \$ 3 each, the sheep \$ 4 each, and the calves \$ 7 each; and the number of calves was equal to the number of pigs and sheep together. How many of each did he buy?

CHAPTER XXIII

THEORY OF EQUATIONS AND SOLUTION OF NUMERICAL HIGHER EQUATIONS

505. When a quadratic equation has the general form

$$x^2 + px = q,$$

we have seen (Art. 344) that

$$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q},$$

in which the unknown quantity is expressed in terms of the general coefficients, giving a formula by which it may be found in all special cases. Solutions of general (*i.e.*, literal) equations of the third and fourth degree are known (see the methods of Cardan and Descartes, Arts. 573 and 576); but these solutions have in many cases difficulties which render them of little practical value, and they are seldom employed. For general equations above the fourth degree no solution is known, *i.e.*, no method of expressing in terms of the literal coefficients the values of the unknown quantity, — indeed, it has been shown by Abel, a Norwegian mathematician, that it is impossible thus to express, algebraically, the values of the unknown quantity. So soon, however, as numerical values are assigned to the coefficients, the real roots may in all cases readily be found, either exactly or to any required degree of approximation.

506. A Root of an Equation is a quantity which, substituted for the unknown quantity, satisfies the equation. It may be commensurable, incommensurable, or imaginary.

507. Commensurable Numbers are such as can be exactly expressed in the decimal notation. They are integers, common

fractions, terminating decimals, and repeating decimals (the latter, as shown in ex. 25, page 205, being reducible to equivalent common fractions).

508. Incommensurable Numbers are such as cannot be exactly expressed in the decimal notation.

Thus, $\sqrt{2}$, $2 \pm \sqrt{3}$, $\sqrt[3]{5}$ are incommensurable numbers.

509. A Multiple Root is a root that occurs more than once in an equation. If it occurs twice, it is called a *double root;* if three times, a *triple root;* if four times, a *quadruple root,* etc.

Thus, from $x^5 - 4x^4 + x^3 + 10x^2 - 4x - 8 = 0$, we have (Art. 101)

$$(x-2)(x-2)(x-2)(x+1)(x+1) = 0,$$

giving (Art. 348) 2 as a triple root and -1 as a double root.

We may say that an equation has the same root repeated several times and call it a **Multiple Root**, or that it has several roots of the same value and call them **Equal Roots**.

510. An equation is said to be in the **Normal** or **Typical Form** when the exponents are all positive integers, the coefficient of the first term (highest power) is 1, and the other coefficients are all integers.

REDUCTION TO THE NORMAL FORM

511. Theorem. *Every algebraical equation having rational coefficients can be transformed into an equation of the normal form in terms of a new unknown quantity which is a known function of the original unknown quantity.*

DEM. *1st. To make the exponents positive, the equation is multiplied by the unknown quantity with a positive exponent equal numerically to the largest negative exponent.* This neither destroys the equality of the members nor changes the value of the unknown quantity.

2d. To make the exponents integral, each exponent is multiplied by the **l. c. m.** *of the denominators of the fractional exponents, and the unknown quantity is replaced by another.* If the equation is in x, and if m is the **l. c. m.** of the denominators of the fractional

exponents, the transformation consists simply in substituting y^m for x. While this changes the roots, the roots of the original equation are known functions of the roots of the transformed equation, viz., the mth power of them.

3d. To make the coefficients integral, and the first one, that of y^n, unity, we divide the equation by the coefficient of y^n, multiply the coefficient of y^{n-1} by k, that of y^{n-2} by k^2, that of y^{n-3} by k^3, and so on, replacing y by z, and then take k of such value as will make each coefficient integral. After the operations of 1st and 2d, and dividing by the coefficient of the highest power, the equation, if arranged, has the form

$$y^n+\frac{a}{a'}y^{n-1}+\frac{b}{b'}y^{n-2}+\frac{c}{c'}y^{n-3}+\cdots\frac{l}{l'}=0.$$

Substituting $y=\frac{z}{k}$, this becomes

$$\frac{z^n}{k^n}+\frac{a}{a'}\frac{z^{n-1}}{k^{n-1}}+\frac{b}{b'}\frac{z^{n-2}}{k^{n-2}}+\frac{c}{c'}\frac{z^{n-3}}{k^{n-3}}+\cdots\frac{l}{l'}=0. \qquad (1)$$

Multiplying by k^n,

$$z^n+\frac{ak}{a'}z^{n-1}+\frac{bk^2}{b'}z^{n-2}+\frac{ck^3}{c'}z^{n-3}+\cdots\frac{lk^n}{l'}=0. \qquad (2)$$

Such value of k may now be taken (it should be the least possible) as will remove the denominators.

We now have $\qquad x=y^m=\left(\frac{z}{k}\right)^m,$

and the roots of the original equation become known when the roots of the equation in z become known.

512. Cor. *From* (1) *and* (2) *it is seen that to obtain an equation whose roots are k times those of a given equation we have but to multiply the coefficient of the 2d term by k, that of the 3d by k^2, that of the 4th by k^3, and so on to the absolute term, which is multiplied by k^n.*

EXAMPLES CXXV

Transform the following into equations of the normal form and express x in terms of the new unknown quantity:

1. $12\,x^{\frac{1}{3}} - 3\,x^{-\frac{1}{6}} - 8\,x^{\frac{1}{2}} + 10\,x^{\frac{1}{6}} - 3\,x^{-\frac{1}{3}} - 10 = 0.$

SOLUTION. Multiplying by $x^{\frac{1}{3}}$,

$$12\,x^{\frac{2}{3}} - 3\,x^{\frac{1}{6}} - 8\,x^{\frac{5}{6}} + 10\,x^{\frac{1}{2}} - 3 - 10\,x^{\frac{1}{3}} = 0.$$

Multiplying the exponents by 6 and replacing x by y,

$$12\,y^4 - 3\,y - 8\,y^5 + 10\,y^3 - 3 - 10\,y^2 = 0.$$

Arranging and dividing by -8,

$$y^5 - \tfrac{3}{2}\,y^4 - \tfrac{5}{4}\,y^3 + \tfrac{5}{4}\,y^2 + \tfrac{3}{8}\,y + \tfrac{3}{8} = 0.$$

Introducing k as indicated in the 3d part of the demonstration and replacing y by z,

$$z^5 - \frac{3\,k}{2}z^4 - \frac{5\,k^2}{4}z^3 + \frac{5\,k^3}{4}z^2 + \frac{3\,k^4}{8}z + \frac{3\,k^5}{8} = 0.$$

Making $k = 2$,

$$z^5 - 3\,z^4 - 5\,z^3 + 10\,z^2 + 6\,z + 12 = 0,$$

in which

$$x = y^6 = \left(\frac{z}{2}\right)^6.$$

By Art. 363, it is found that -2 is a root of this equation; hence

$$y = \frac{-2}{2} = -1, \text{ and } x = (-1)^6 = 1.$$

Therefore 1 is the corresponding root of the original equation.

2. $\dfrac{2}{x} - 3\,x + \dfrac{4}{5}x^{\frac{1}{2}} - 2 = 0.$

3. $\dfrac{x^2+1}{1-x^{-2}} = \dfrac{2}{3}(1 + x^{-2}).$

4. $\dfrac{1 - x^{\frac{2}{3}}}{1 + x^{-1}} = \dfrac{x^{-1}+3}{x+2}.$

5. $2\sqrt{1 - x^2} = 4 - 3\,x^{-\frac{1}{3}}.$

6. $3\,x^5 - 2\,x^4 - \dfrac{2}{3}x^3 + 5\,x^2 - \dfrac{4}{9}x + 1 = 0.$

7. $3\,x^{-\frac{4}{3}} + \dfrac{3}{5}x^{-\frac{1}{2}} + \dfrac{4}{x^3} - \dfrac{1}{5}x^{-\frac{5}{2}} + \dfrac{1}{x} - \dfrac{2}{5}x^2 = \dfrac{1}{10}x^{-4}.$

TEST FOR ROOTS

513. Theorem. *If $f(x)$ be divided by $x - a$, the last remainder will be $f(a)$;* i.e., *will be what $f(x)$ becomes when a is substituted for x.*

DEM. Let $\phi(x)$ be the quotient and r the last remainder. Then, by the laws of division,

$$f(x) = (x - a)\,\phi(x) + r.$$

As this equation is true for all values of x, it is true when $x = a$. Substituting a for x, we have

$$f(a) = r.$$

514. Sch. It follows from the last theorem that the short method of dividing by $x - a$, given in Art. 79, becomes a short method of substituting any number for x.

The operation of substituting a value for x is often called *evaluating*.

EXAMPLES CXXVI

Find what the following become when the indicated values of x are substituted:

1. $x^5 - 2x^4 - 4x^3 + 3x^2 - 5x + 6$ for $x = 3$.

Solution. By the method of division given in Art. 79, we have

$$\begin{array}{cccccc|l} x^5 & -\,2x^4 & -\,4x^3 & +\,3x^2 & -\,5x & +\,6 & \underline{3} \\ & 1 & -1 & 0 & -5 & -9 & \end{array}$$

Therefore, by Art. 513, this polynomial becomes -9 when 3 is substituted for x.

2. $x^4 + x^3 - 17x^2 - 10x + 11$ for $x = 4$.

3. $x^5 - 8x^4 + 18x^3 - 12x^2 - 14x - 4$ for $x = 5$.

4. $3x^6 - 4x^5 - 6x^3 - 4x^2 - 10x + 23$ for $x = 2$.

Operation

$$\begin{array}{ccccccc|l} 3x^6 & -\,4x^5 & +\,0x^4 & -\,6x^3 & -\,4x^2 & -\,10x & +\,23 & \underline{2} \\ & 2 & 4 & 2 & 0 & -10 & 3 & \end{array}$$

Therefore this polynomial becomes 3 when 2 is substituted for x. The same operation shows (Art. 79) that when the polynomial is divided by $x - 2$, the quotient is $3x^5 + 2x^4 + 4x^3 + 2x^2 - 10$, with a remainder of 3.

5. $5x^5 + 12x^4 + 6x^3 - 3x^2 - 13x + 9$ for $x = -2$.

6. $4x^5 - 17x^4 + 18x^2 - 20$ for $x = 4$.

7. $x^6 - 8x^5 + 10x^4 + 40x^3 - 71x^2 - 32x + 60$ for $x = 5$.

515. Theorem. *If $f(x)$ is divisible by $x - a$, a is a root of $f(x) = 0$; and, conversely, if a is a root of $f(x) = 0$, $f(x)$ is divisible by $x - a$.*

DEM. By Art. 513 if $f(x)$ be divided by $x-a$, the remainder is $f(a)$. Now, if the remainder is 0, $f(a)=0$, and the equation is satisfied.

Again, using the notation of Art. 513,

$$(x-a)\phi(x)+r=f(x).$$

Now if a is a root, $f(x)$ is 0 and $x-a$ is 0, and the equation becomes

$$0+r=0.$$

Hence, the remainder being 0, $f(x)$ is divisible by $x-a$.

516. **Cor.** *If a is a root of $f(x)=0$, a is a factor of the absolute term.*

If a is a root, there is no remainder when $f(x)$ is divided by $x-a$; therefore, as readily appears from the process of division, a is a factor of the absolute term.

NUMBER AND CHARACTER OF THE ROOTS

517. **Theorem.** *An equation in the normal form cannot have a fractional root, and hence the real roots of such an equation are either integral or incommensurable.*

DEM. Suppose $\frac{s}{t}$, a simple fraction in its lowest terms, to be a root of the normal equation

$$x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+\cdots L=0.$$

Substituting this value of x, we have

$$\frac{s^n}{t^n}+A\frac{s^{n-1}}{t^{n-1}}+B\frac{s^{n-2}}{t^{n-2}}+C\frac{s^{n-3}}{t^{n-3}}+\cdots+L=0.$$

In this equation all the terms except the first are integral, and the first is a simple fraction in its lowest terms, s and t being prime to each other. But the sum of a simple fraction in its lowest terms and a series of integers cannot be 0. Hence x cannot equal $\frac{s}{t}$, a fraction.

518. SCH. It is readily seen that the above reasoning does not apply if the coefficient of the first term is other than 1; for in

that case the coefficient might contain t, making the first term integral instead of fractional when $\frac{s}{t}$ is substituted for x.

519. Theorem. *An equation of the nth degree has n roots and no more.**

Dem. Let a be a root of $f(x)=0$. Then $f(x)$ is divisible by $x-a$ (Art. 515). Representing the quotient by $\phi(x)$, we have

$$(x-a)\phi(x)=0. \tag{1}$$

This is satisfied not only for $x-a=0$, but also for $\phi(x)=0$ (Art. 348), and the degree of $\phi(x)$ is one lower than that of $f(x)$. If b is a root of $\phi(x)=0$, it is also a root of $f(x)=0$, as seen from (1). Dividing $\phi(x)$ by $x-a$, the degree will again be diminished by 1. This process may be continued for n divisions and no more. Hence $f(x)=0$ has n roots and no more.

520. Cor. 1. *To form an equation whose roots shall be $a, b, c, \cdots l$, we have*

$$f(x)=(x-a)(x-b)(x-c)\cdots(x-l)=0;$$

i.e., *to form an equation having any roots whatever, we have but to subtract these roots from x and place the product of the remainders equal to* 0.

521. Cor. 2. *The equation $f(x)=0$ may have 2, 3, or even n equal roots, as in the above demonstration we may have*

$$a=b,\ a=b=c, \text{ or } a=b=c=\cdots l.$$

522. Theorem. *If an equation having only rational coefficients has imaginary or quadratic surd roots, these roots enter in conjugate pairs* (Art. 221).

Dem. If $\beta\sqrt{-1}$ is a root of $f(x)=0$, $x-\beta\sqrt{-1}$ is a factor of $f(x)$ (Art. 515). Now the only factor by which $x-\beta\sqrt{-1}$ can be multiplied to produce a real and rational quantity is $x+\beta\sqrt{-1}$, giving $x^2+\beta^2$ for the product. Hence (Art. 515) $-\beta\sqrt{-1}$ is also a root.

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* It is here assumed that every equation has at least one root. This has been proved by Argand, Gauss, Cauchy, Clifford, and others.

Again, if $\alpha + \beta\sqrt{-1}$ is a root of $f(x) = 0$, $x - (\alpha + \beta\sqrt{-1})$ is a factor of $f(x)$; hence to give a real and rational product, $x - (\alpha - \beta\sqrt{-1})$ must also be a factor, giving $\alpha - \beta\sqrt{-1}$ as a root, and $(x - \alpha)^2 + \beta^2$ as the product of the two factors.

It may be shown in the same way that if $\beta\sqrt{\gamma}$ is a root of $f(x) = 0$, $-\beta\sqrt{\gamma}$ must also be a root; and that if $\alpha + \beta\sqrt{\gamma}$ is a root of $f(x) = 0$, $\alpha - \beta\sqrt{\gamma}$ must also be a root.

523. Cor. 1. *An equation of odd degree has at least one real root.*

524. Cor. 2. *All the roots of an equation of even degree can be imaginary only when the absolute term is positive.*

For, as seen above, the product of the factors obtained by subtracting conjugate imaginaries from x always gives a positive absolute term.

525. Cor. 3. *An equation of even degree whose absolute term is negative has at least two real roots, and these have opposite signs.*

526. Theorem. *Changing the signs of the terms containing the odd powers, or the even powers, or, if the equation is complete, the alternate terms, changes the signs of the roots of an equation;* i.e., *the positive roots of $f(-x) = 0$ are the negative roots of $f(x) = 0$.*

Dem. Let $a, b, c, \cdots l$ be the roots of an equation $f(x) = 0$; then (Art. 520)

$$f(x) = (x - a)(x - b)(x - c) \cdots (x - l) = 0.$$

Replacing x by $-x$, we have

$$f(-x) = (-x - a)(-x - b)(-x - c) \cdots (-x - l) = 0,$$

or, removing -1 from each factor,

$$f(-x) = (-1)^n(x + a)(x + b)(x + c) \cdots (x + l) = 0.$$

By equating each of these binomial factors with 0 (Art. 348), the roots are seen to be $-a, -b, -c, \cdots -l$; *i.e.*, changing the sign of x changes the signs of the roots. But changing the sign of x changes the signs of the terms containing the odd powers, and only these.

Changing the signs of the terms containing the even powers also changes the signs of the roots, since this is the same as multiplying or dividing both members by -1 after changing the signs of the terms containing the odd powers.

527. Sch. The absolute term may be regarded as the coefficient of x^0, and it must be noted that 0 is an even number.

528. A Permanence is a succession of two like signs in the consecutive terms of a polynomial; a **Variation**, of two unlike signs.

Thus, in $x^6 - 2x^5 - 4x^4 + 5x^3 + 3x^2 + x - 4$, the signs being $+ - - + + + -$, there are three *permanences* and three *variations.*

529. Descartes' Rule of Signs. *An equation in the form $f(x) = 0$, whether complete or incomplete, cannot have more positive roots than the number of variations in $f(x)$, nor more negative roots than the number of variations in $f(-x)$.*

Verification. Let the signs and missing terms of $f(x)$ be

$$+ + 0 - + 0\,0 - - - +,$$

giving 4 variations.

Let us now introduce a positive root by multiplying by x minus this root (Art. 515), writing only the signs in the operation, as follows:

$$\begin{array}{cccccccccccc}
+ & + & 0 & - & + & 0 & 0 & - & - & - & + & \\
+ & - & & & & & & & & & & \\
\hline
+ & + & 0 & - & + & 0 & 0 & - & - & - & + & \\
 & - & - & 0 & + & - & 0 & 0 & + & + & + & - \\
\hline
+ & \pm & - & - & + & - & 0 & - & \mp & \mp & + & -
\end{array}$$

The ambiguous signs $\pm$ and $\mp$ are placed where the sign to be used must be determined by the relative numerical values of the coefficients.

Counting, now, the fewest number of variations that can occur, by taking that one of the ambiguous signs that will give, with the preceding sign, a permanence, we have 5 variations, a variation being necessarily introduced at the end. Thus each time a positive root is introduced by multiplying by x minus that root, at least one variation is added. Hence an equation cannot have more positive roots than the number of variations.

Again the positive roots of $f(-x)=0$ are the negative roots of $f(x)=0$ (Art. 526). As just shown, $f(-x)=0$ cannot have more positive roots than the number of variations in $f(-x)$. Hence $f(x)=0$ cannot have more negative roots than the number of variations in $f(-x)$.

530. Cor. 1. *An equation whose terms are all positive has no positive roots.*

531. Cor. 2. *An equation whose terms of even power are all of one sign and whose terms of odd power are all of the contrary sign has no negative roots. Why?*

Hence a complete equation whose terms are alternately + and − has no negative roots.

532. Cor. 3. *An equation having only even powers, and these all of the same sign, has no real roots. An equation having only odd powers, and these all of the same sign, has no real root except* 0.

533. Note. The truth of the above corollaries is readily seen independently of Descartes' Rule of Signs.

534. Cor. 4. *An incomplete equation has at least as many imaginary roots as the difference between the degree of the equation and the total number of variations in $f(x)$ and $f(-x)$.*

For the total number of roots is the same as the degree of the equation, and there cannot be more real roots than the total number of variations in $f(x)$ and $f(-x)$.

535. Cor. 5. *A complete equation cannot have more negative roots than its number of permanences.*

The total number of variations and permanences is one less than the number of terms, *i.e.*, the same as the degree of the equation or the total number of roots. Now there cannot be more positive roots than the number of variations, and hence not more negative roots than the number of permanences.

536. Note. Descartes' Rule and the corollaries deduced from it are useful only in preventing, in some cases, a fruitless search for roots of a particular sign after all of that sign have been found, or for the situation of roots after the situation of all the real ones has been found.

EXAMPLES CXXVII

Determine the greatest number of positive and negative, and the least number of imaginary, roots in the following:

1. $x^6 + 8x^3 + 3x - 6 = 0.$

Solution. By Art. 525 this equation has 2 real roots, 1 positive and 1 negative. Since there is but 1 variation, the equation cannot have more than 1 positive root. Changing the signs of the terms containing the odd powers of x, we have

$$x^6 - 8x^3 - 3x - 6 = 0.$$

Since this has but 1 variation, the original equation cannot have more than 1 negative root. But the equation, being of the 6th degree, has 6 roots; hence it has 4 imaginary roots.

2. $x^4 + 12x^2 + 5x - 10 = 0.$

3. $3x^5 + 4x + 5 = 0.$

4. $x^5 - 7x^2 - 6 = 0.$

5. $x^6 - 7x^2 - x + 3 = 0.$

6. $2x^6 - 4x^5 - 5x^3 + 3x^2 - 5 = 0.$

7. $x^6 + 7x^5 - 4x^4 - 5x^3 - 3x - 6 = 0.$

8. $x^6 - 2x^5 + x^4 - x^2 - 1 = 0.$

9. $x^n - 1$, when n is odd.

10. $x^n - 1$, when n is even.

11. $x^n + 1$, when n is odd.

12. $x^n + 1$, when n is even.

SOLUTION FOR COMMENSURABLE ROOTS

537. The process of finding the roots of an equation when all the roots, or all but two of them, are integral, has been fully explained in Chapter XVI. We now see from Art. 513 that the short process of factoring there used is also a short process of evaluating for, or substituting, any particular value of the unknown quantity. A few additional examples are here given for further practice in that important process. In applying it the student should keep in mind the following:

1. When the equation is in the normal form, only factors of the absolute term need be tried (Art. 516).

2. It will be found expedient to try the small positive factors first. Before writing down the various sums, it is best, if the coefficients are not too large, to run through mentally with a factor of the last term and see whether the last addition gives 0.

3. If all the terms are positive, no positive numbers need be tried (Art. 102); and if at any stage in the operation of evaluating, all the sums become positive, no more positive numbers need be tried.

4. When all but two of the roots have been found, the remaining two, whatever their values, may be found by solving the resulting quadratic. The last four roots, even if surd or imaginary, may be found if the alternate terms of the depressed equation are 0, since this depressed equation would then have the quadratic form (Art. 359).

EXAMPLES CXXVIII

Find all the roots of the following:

1. $x^6 - 4x^5 - 6x^4 + 40x^3 - 31x^2 - 36x + 36 = 0.$

OPERATION

$$\begin{array}{ccccccc|l}
x^6 - 4x^5 & - 6x^4 & + 40x^3 & - 31x^2 & - 36x & + 36 & & \underline{1} \\
-3 & -9 & 31 & 0 & -36 & 0 & & \underline{2} \\
-1 & -11 & 9 & 18 & 0 & & & \underline{2} \\
1 & -9 & -9 & 0 & & & & \underline{3} \\
4 & 3 & 0 & & & & & \underline{-1} \\
3 & 0 & & & & & & \underline{-3} \\
0 & & & & & & &
\end{array}$$

Or the last two roots may be found by solving the quadratic

$$x^2 + 4x + 3 = 0, \text{ giving } x = -2 \pm 1 = -1 \text{ and } -3.$$

2. $x^3 - x^2 - 14x + 24 = 0.$ **3.** $x^3 - 7x + 6 = 0.$

4. $x^3 - 15x^2 + 74x - 120 = 0.$ **5.** $x^3 - 7x^2 - 5x + 35 = 0.$

6. $x^4 - 10x^3 + 24x^2 + 10x - 25 = 0.$

7. $x^4 - 10x^3 + 20x^2 + 10x - 21 = 0.$

8. $x^5 - 5x^4 - 13x^3 + 65x^2 + 36x - 180 = 0.$

9. $x^5 + 2x^4 - 3x^3 - 3x^2 + 2x + 1 = 0.$

10. $x^5 - 6x^4 + 15x^3 - 26x^2 + 36x - 24 = 0.$

11. $x^5 + 2x^4 - 9x^3 + 14x - 8 = 0.$

12. $x^5 - 5x^3 + 6x^2 - 14x + 12 = 0.$

13. $x^5 - 15x^4 + 85x^3 - 225x^2 + 274x - 120 = 0.$

14. $x^5 - 2x^4 - 6x^3 + 12x^2 + 9x - 18 = 0.$

15. $x^5 - x^4 + 4x^3 - 4x^2 + 4x - 4 = 0.$

16. $4x^5 - 4x^4 - 31x^3 + 10x^2 + 39x - 18 = 0.$

SUGGESTION. When the coefficient of the first term is not 1, first find all the integral roots, and then, if necessary, transform the depressed equation to the normal form.

17. $6x^5 - 19x^4 + 13x^3 + 13x^2 - 19x + 6 = 0.$

18. $x^6 - 3x^5 - 3x^4 + 15x^3 - 6x^2 - 12x + 8 = 0.$

19. $x^6 - 4x^5 - 4x^4 + 20x^3 - x^2 - 16x + 4 = 0.$

20. $x^7 - 2x^6 - 10x^5 + 28x^4 + 5x^3 - 74x^2 + 76x - 24 = 0.$

21. $12x^6 - 20x^5 - 85x^4 + 92x^3 + 97x^2 - 132x + 36 = 0.$

EQUATIONS WITH ONLY EVEN OR ONLY ODD POWERS

538. When an equation contains only even powers of x, we may regard x^2 as the unknown quantity and, without supplying the missing odd powers, proceed as above and then extract the square root of the numbers which reduce the function to 0. This is of great importance in finding the roots of such an equation when they are monomial surds or monomial imaginaries.

An equation containing only odd powers can, after dividing by x (for in that case there could be no absolute term), be treated in the same way.

EXAMPLES CXXIX

Find all of the roots of the following:

1. $x^6 - 14x^4 + 49x^2 - 36 = 0.$

SOLUTION. This may be written

$$(x^2)^3 - 14(x^2)^2 + 49(x^2) - 36 = 0.$$

Regarding x^2 as the unknown quantity, there are no missing terms. We may, therefore, proceed as follows:

$$\begin{array}{cccc|l} (x^2)^3 - 14(x^2)^2 & + 49(x^2) & - 36 & & \underline{1} \\ -13 & 36 & 0 & & \underline{4} \\ -9 & 0 & & & \underline{9} \\ 0 & & & & \end{array}$$

Hence the factors are x^2-1, x^2-4, x^2-9, and the roots are ± 1, ± 2, ± 3.

2. $x^8 - 3x^6 - 15x^4 + 19x^2 + 30 = 0.$

OPERATION

$$\begin{array}{ccccc|l} x^8 & - 3x^6 & - 15x^4 & + 19x^2 & + 30 & \underline{2} \\ & -1 & -17 & -15 & 0 & \underline{5} \\ & 4 & 3 & 0 & & \underline{-1} \\ & 3 & & & & \underline{-3} \\ & 0 & & & & \end{array}$$

Hence the factors are $x^2 - 2$, $x^2 - 5$, $x^2 + 1$, $x^2 + 3$, and the roots are $\pm\sqrt{2}$, $\pm\sqrt{5}$, $\pm\sqrt{-1}$, $\pm\sqrt{-3}$.

3. $x^6 - 7x^4 + 14x^2 - 8 = 0.$ **4.** $x^6 - 7x^4 + 16x^2 - 12 = 0.$

5. $x^6 - 10x^4 + 31x^2 - 30 = 0.$ **6.** $x^6 - 13x^4 + 35x^2 + 49 = 0.$

7. $x^6 - 16x^4 + 85x^2 - 150 = 0.$ **8.** $x^6 - 15x^4 + 74x^2 - 120 = 0.$

9. $x^6 - 7x^4 + 7x^2 + 15 = 0.$ **10.** $x^7 - 6x^5 + 3x^3 + 10x = 0.$

11. $x^8 - 14x^6 + 65x^4 - 124x^2 + 84 = 0.$

12. $x^8 - 5x^6 - 10x^4 + 20x^2 + 24 = 0.$

13. $x^9 - 9x^7 + 14x^5 + 36x^3 - 72x = 0.$

14. $x^8 + 11x^6 + 41x^4 + 61x^2 + 30 = 0.$

15. $x^{10} - 4x^8 - 3x^6 + 22x^4 - 4x^2 - 24 = 0.$

16. $x^5 - 3x^4 - 7x^3 + 21x^2 + 10x - 30 = 0.$

17. $x^7 - 3x^6 - 10x^5 + 30x^4 + 31x^3 - 93x^2 - 30x + 90 = 0.$

RELATION OF ROOTS TO COEFFICIENTS

539. Prob. *To find the relation between the roots and the coefficients of an equation.*

SOLUTION. If a, b, c, are the three roots of an equation, we have (Art. 520)

$$f(x) = (x - a)(x - b)(x - c) = 0,$$

or $$x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc = 0.$$

If a, b, c, d, are the four roots of an equation, we have

$$f(x) = (x - a)(x - b)(x - c)(x - d) = 0,$$

or $$x^4 - (a + b + c + d)x^3 + (ab + ac + ad + bc + bd + cd)x^2$$
$$- (abc + abd + acd + bcd)x + abcd = 0.$$

Generalizing, we have the following theorem:

The coefficient of x^{n-1} *is the sum of the roots with their signs changed.*

The coefficient of x^{n-2} *is the sum of the products of the roots taken two and two.*

The coefficient of x^{n-3} *is the sum of the products of the roots, with their signs changed, taken three and three, and so on.*

The absolute term is the product of all the roots with their signs changed.

540. In forming an equation from its roots the student will find it safer and more expeditious to proceed as directed in Art. 349.

Thus, to produce the equation whose roots are 4, 3, 2, -2, -1, we have

$$f(x) = (x - 4)(x - 3)(x - 2)(x + 2)(x + 1) = 0.$$

The product of the first two factors is found by the process of Art. 60 to be $x^2 - 7x + 12$. We then proceed as follows:

$$\begin{array}{rrrrl}
(-2) & 1 - 7 & 12 & & \\
(2) & -9 & 26 & -24 & \\
(1) & -7 & 8 & 28 - 48 & \\
 & -6 & 1 & 36 - 20 - 48 &
\end{array}$$

Hence the equation is

$$x^5 - 6x^4 + x^3 + 36x^2 - 20x - 48 = 0.$$

If some of the roots are imaginaries or quadratic surds, and hence occur in conjugate pairs, the remainders should be multiplied together in pairs, so as to make use of the theorem for the product of the sum and difference of two quantities.

Thus, to produce the equation whose roots are $2 \pm \sqrt{3}$, $1 \pm \sqrt{-1}$, we have

$$f(x) = (x - 2 - \sqrt{3})(x - 2 + \sqrt{3})(x - 1 - \sqrt{-1})(x - 1 + \sqrt{-1}) = 0,$$

or
$$f(x) = (x^2 - 4x + 1)(x^2 - 2x + 2) = 0.$$

The indicated multiplication should now be performed by the process of Art. 56.

EXAMPLES CXXX

Produce the equations whose roots are the following:

1. $1, 2, 3, -4$.
2. $2, 5, -1, -3$.
3. $1, 1, 2, \pm\sqrt{2}$.
4. $1, -1, \pm\sqrt{2}, \pm\sqrt{3}$.
5. $1, 2, 3, 4, -1, -2$.
6. $\pm\sqrt{2}, \pm\sqrt{5}, \pm\sqrt{-1}$.
7. $\frac{1}{2}, \frac{3}{2}, -1, -3, -4$.
8. $\frac{1}{2}, \frac{2}{3}, -1, \pm\sqrt{3}$.
9. $2, -2, 1 \pm\sqrt{2}, 2 \pm\sqrt{-1}$.
10. $3 \pm\sqrt{5}, 2 \pm\sqrt{2}, 3 \pm\sqrt{-3}$.

SITUATION OF ROOTS

541. Theorem. *If $f(x)$ has opposite signs when two different numbers are substituted for x, an odd number of real roots lie between the substituted numbers; while if $f(x)$ has the same signs for both numbers, either no real roots, or an even number of real roots, lie between them.*

DEM. Let a, b, c, etc., in the order of their values, a being least, be the real roots of $f(x) = 0$, and let $\phi(x)$ be the product of the remainders obtained by subtracting the imaginary roots, if any, from x; then (Art. 520),

$$f(x) = (x - a)(x - b)(x - c) \cdots (\phi(x)) = 0.$$

If any number be substituted for x, all the above factors containing roots less than the substituted number will be $+$, and all containing roots greater than the substituted number will be $-$. If now a second number be substituted for x, only those factors containing roots between the substituted numbers will change sign. Hence, since $\phi(x)$ is always $+$ (Art. 522), a change of sign of the product would indicate that an odd number of real roots lie

between the substituted numbers; while no change of sign of the product would indicate that no real roots or an even number of real roots lie between the substituted numbers.

Or the truth of the theorem may be seen thus:

All values < 0 are $-$ and all values > 0 are $+$. Now if one value of x makes $f(x) -$, *i.e.*, too small, and another value of x makes $f(x) +$, *i.e.*, too large, there must be a value of x between these two that will make $f(x)$ 0, *i.e.*, satisfy the equation.

In passing from a value too small to a value too large, *i.e.*, from $-$ to $+$, or *vice versa*, $f(x)$ might pass through 0 three, five, or some other odd number of times; but if for two different values of x, $f(x)$ has the same sign, it is because $f(x)$ has either not passed through 0 at all, or passed through an even number of times, leaving it on the same side of 0.

542. Cor. *Whenever the substitution of two consecutive numbers gives opposite signs to $f(x)$, the numerically smaller of the two numbers is the integral part of the root that lies between them.*

543. Sch. 1. While we may find the situation of negative roots by evaluating $f(x)$ for negative numbers, it is a little more convenient to evaluate $f(-x)$ for positive numbers (Art. 526). Besides, as will be seen later, in approximating beyond the integral part the value of a negative root, it is necessary to employ $f(-x)$.

544. Sch. 2. By taking the value of x large enough the first term of $f(x)$ may be made larger than any or all of the other terms. Hence, if a series of minus signs result from the substitution of larger and larger numbers, the situation of a root will be found by taking still larger numbers, as the function must eventually become plus, the sign of the first term.

EXAMPLES CXXXI

Find the situation of all the real roots of the following:

1. $x^4 - 2x^3 - 13x^2 + 22x + 22 = 0.$

Solution. Since $f(x)$ has but two variations, the equation cannot have more than two positive roots (Art. 529). Substituting for x, by Art. 513,

the values 0, 1, 2, 3, etc., we have for $f(x)$ the signs given in the margin, showing that the positive roots lie between 2 and 3, 3 and 4.

x	$f(x)$	x	$f(-x)$
0	+	0	+
1	+	1	−
2	+	2	−
3	−	3	−
4	+	4	+

Changing the signs of the alternate terms of $f(x)$, we have

$$f(-x)=x^4+2x^3-13x^2-22x+22=0.$$

The positive roots of this equation are seen to lie between 0 and 1, 3 and 4; therefore (Art. 526) the negative roots of the original equation lie between 0 and -1, -3 and -4.

2. $x^3-3x^2-4x+11=0.$　3. $x^3-3x^2-10x+5=0.$

4. $x^3+6x^2+2x-1=0.$　5. $x^3-6x^2+2x+2=0.$

6. $x^4-4x^3-3x+23=0.$

Sug. Since $f(x)$ has but two variations, the equation cannot have more than two positive roots (Art. 529).

Since all the signs of $f(-x)$ are $+$, $f(-x)=0$ can have no positive roots (Art. 529); hence $f(x)=0$ can have no negative roots. It follows that two of the four roots are imaginary.

7. $x^3-2x^2-10=0.$　8. $x^3-3x-1=0.$

9. $x^3-2x^2-6x-2=0.$　10. $x^3-5x^2-4=0.$

11. $x^3+x^2-10=0.$

Solution. The position of the one positive root is easily found.

Changing the signs of the terms containing the even powers (Art. 526), we have

$$f(-x)=x^3-x^2+10=0.$$

This function can become 0 only by having the single negative term numerically equal to the sum of the two positive terms. Now, if $x<1$, the 10 alone, without the x^3, is more than enough to counterbalance the negative term; while if $x>1$, the x^3 alone without the 10, is more than enough to counterbalance the negative term.

Hence this equation can have no positive roots, and the original equation no negative roots.

12. $x^3+x^2+x-50=0.$　13. $x^5+x^4+x^2-20=0.$

14. $8x^3-36x^2+46x-15=0.$　15. $x^4-10x^3+31x^2-30x+6=0.$

16. $x^4-6x^3+4x^2+18x-21=0.$

SOLUTION. The one variation of $f(-x)$ shows that the equation cannot have more than one negative root. A negative root is readily located between -1 and -2.

Evaluating $f(x)$ for positive numbers, the results are as in the margin; locating one root between 4 and 5. As all numbers >5 give positive results, the other two roots must either both be imaginary, or both be situated between two consecutive numbers. It is observed that both 1 and 2 very nearly satisfy the equation (causing $f(x)$ to differ little from 0), suggesting that two roots may lie between these numbers. Evaluating for 1.1, 1.2, 1.3, etc., we find changes of signs between 1.5 and 1.6, and between 1.7 and 1.8, thus locating the other two roots.

x	$f(x)$
0	-21
1	-4
2	-1
3	-12
4	-13
5	$+$

17. $2x^3 - 3x^2 - 6x + 9 = 0.$

SUG. Proceeding as in the preceding example, it is rendered probable that there are two roots between 1 and 2. Evaluating for the tenths, 1.5 is found to be a root. The best way to find the remaining two roots is to depress the equation and solve the resulting quadratic.

Or this equation may be transformed into one of the normal form, in which form it has one integral root.

18. $x^4 - 2x^3 - 16x^2 + 24x + 48 = 0.$

19. $x^4 - 4x^3 - x^2 + 12x - 6 = 0.$

20. $x^5 - 6x^4 + 36x^2 - 25x - 30 = 0.$

SUG. Remove the factor containing the integral root and use the depressed equation.

21. $x^5 - 5x^4 + 3x^3 + 16x^2 - 26x + 11 = 0.$

22. $x^5 - 2x^3 - x^2 - 3x - 1 = 0.$

23. $x^4 - 10x^3 + 33x^2 - 40x + 14 = 0.$

MULTIPLE ROOTS

545. Theorem. *If an equation $f(x) = 0$ has equal roots, they are the roots, and the only roots, of the equation formed by placing equal to 0 the highest common divisor of $f(x)$ and its first differential coefficient.*

DEM. Let a be one of the m equal roots of $f(x) = 0$. Then $(x-a)^m$ is a factor of $f(x)$. If we represent the product of the other factors by $\phi(x)$, we have

$$f(x) = (x-a)^m \phi(x) = 0. \tag{1}$$

Representing the first differential coefficients of $f(x)$ and $\phi(x)$ by $f'(x)$ and $\phi'(x)$ respectively, we have (Arts. 415 and 420)

$$f'(x) = m(x-a)^{m-1}\phi(x) + (x-a)^m \phi'(x) = 0. \qquad (2)$$

Comparing (2) with (1), we see that $(x-a)^{m-1}$ is the h. c. d. of $f(x)$ and $f'(x)$, and that the equation,

$$(x-a)^{m-1} = 0,$$

has $m-1$ roots equal to a, and has no other roots.

Similarly, if a is one of the m equal roots and b one of the n equal roots of $f(x) = 0$, we shall have

$$(x-a)^{m-1}(x-b)^{n-1} = 0,$$

which has $m-1$ roots equal to a and $n-1$ roots equal to b, and has no other roots.

546. The work of finding the h. c. d. of $f(x)$ and $f'(x)$ is so great that the process should be resorted to only when shorter processes fail. It may be avoided in the following cases:

1. Multiple integral roots are found by the same operation that gives other integral roots (Art. 537).

2. Multiple monomial imaginary and quadratic surd roots in equations containing only even powers or only odd powers are found as in Art. 538.

3. Multiple roots in an equation in which $f(x)$ is a perfect square are found from the equation obtained by extracting the square root.

4. Finally, in obtaining multiple roots, it is never necessary to find the h. c. d. of $f(x)$ and $f'(x)$ for equations below the 6th degree, as seen from the following considerations:

(*a*) If a cubic has a multiple root, we must have either

$$(x-a)^3 = 0 \text{ or } (x-a)^2(x-b) = 0.$$

In both forms, to give rational coefficients, the roots must be commensurable.

(*b*) If a biquadratic has multiple roots we must have one of the forms

$$(x-a)^4 = 0,\ (x-a)^3(x-b) = 0,\ (x-a)^2(x-b)(x-c) = 0,$$

or

$$(x-a)^2(x-b)^2 = 0.$$

In the first three forms, to give rational coefficients, the multiple roots must be commensurable. In the fourth form a and b can be incommensurable if they are numerically equal with opposite signs, giving

$$(x-a)^2\,(x+a)^2=(x^2-a^2)^2=0.$$

In this case $f(x)$ is a perfect square.

(c) If a quintic has multiple roots, we must have one of the forms

$$(x-a)^5=0,\ \ (x-a)^4\,(x-b)=0,\ \ (x-a)^3\,(x-b)\,(x-c)=0,$$

$$(x-a)^3\,(x-b)^2=0,\ \ (x-a)^2\,(x-b)\,(x-c)(x-d)=0,$$

or

$$(x-a)^2\,(x-b)^2\,(x-c)=0.$$

In all but the last of these forms, to give rational coëfficients, the multiple roots must be commensurable. In the last form a and b can be incommensurable if they are numerically equal with opposite signs, giving

$$(x-a)^2\,(x+a)^2\,(x-c)=(x^2-a^2)^2\,(x-c)=0.$$

In this case the commensurable root c may be removed (Art. 515), leaving a perfect square.

EXAMPLES CXXXII

Find the multiple roots of the following:

1. $x^6-4x^5-x^4+16x^3-5x^2-12x-3=0.$

Solution. Since the equation is of even degree and the absolute term $-$, it has at least two real roots, and these have opposite signs (Art. 525). As indicated in the margin, there are no changes of sign except between 1 and 2, and between -1 and -2.

Now, if any of the remaining roots are real, they must be situated either between these same numbers (a change of sign indicates a passage through an odd number of roots), or in pairs between other consecutive numbers. We therefore apply the test for equal roots.

x	$f(x)$	x	$f(-x)$
0	$-$	0	$-$
1	$-$	1	$-$
2	$+$	2	$+$
3	$+$	3	$+$

Obtaining the first differential coefficient, we have

$$f'(x)=6x^5-20x^4-4x^3+48x^2-10x-12.$$

Rejecting the factor 2, finding the h. c. d. of $f(x)$ and $f'(x)$, and placing it equal to 0, we have

$$x^2-2x-1=0,$$

or

$$x=1\pm\sqrt{2}.$$

Therefore $1 + \sqrt{2}$ and $1 - \sqrt{2}$ are each double roots.

Again, since $x^2 - 2x - 1$ is the **h. c. d.** of $f(x)$ and $f'(x)$, $(x^2 - 2x - 1)^2$, or $x^4 - 4x^3 + 2x^2 + 4x + 1$, is a factor of $f(x)$ (Art. 545). The other factor is found by division to be $x^2 - 3$, giving for the other roots $x = \pm\sqrt{3}$. Hence the six roots are $1 \pm \sqrt{2},\ 1 \pm \sqrt{2},\ \pm\sqrt{3}$.

2. $x^4 - 8x^3 + 18x^2 - 8x + 1 = 0.$
3. $x^4 - 12x^3 + 50x^2 - 84x + 49 = 0.$
4. $x^5 - 7x^4 + 8x^3 + 28x^2 - 22x - 48 = 0.$
5. $x^6 - 7x^4 + 16x^2 - 12 = 0.$
6. $x^6 - 2x^5 - 6x^4 + 8x^3 + 12x^2 - 8x - 8 = 0.$
7. $x^7 - 12x^5 + 45x^3 - 50x = 0.$
8. $x^6 - 8x^5 + 12x^4 + 24x^3 - 27x^2 - 16x - 1 = 0.$
9. $x^8 - 9x^6 + 30x^4 - 44x^2 + 24 = 0.$
10. $x^8 - 8x^7 + 16x^6 + 16x^5 - 56x^4 - 32x^3 + 64x^2 + 64x + 16 = 0.$

SOLUTION FOR INCOMMENSURABLE ROOTS

Horner's Method

547. Having explained how to find the situation and, consequently, the initial figures of the real roots of an equation, we now proceed to exhibit a process for finding the remaining parts of such roots, exactly if commensurable, and to any required degree of approximation if incommensurable. This process is called **Horner's Method**, having been published by W. G. Horner, of Bath, England, in 1819. The method is based on the next two theorems.

548. Theorem. *If the first member of an equation in the form* $f(x) = 0$ *be divided by* $x - a$, *then the integral part of the quotient be again divided by* $x - a$, *and so on, the successive remainders will be, in inverse order, the coefficients of an equation whose roots are less by* a *than those of the given equation.*

Dem. Let the given equation be

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \cdots Jx^2 + Kx + L = 0. \tag{1}$$

Let $x = x_1 + a$. Then by substitution

$$A(x_1 + a)^n + B(x_1 + a)^{n-1} + C(x_1 + a)^{n-2} + \cdots J(x_1 + a)^2$$
$$+ K(x_1 + a) + L = 0.$$

If we should perform the indicated operations and arrange with reference to x_1, there would result an equation of the form

$$Ax_1^n + B_1x_1^{n-1} + C_1x_1^{n-2} + \cdots J_1x_1^2 + K_1x_1 + L_1 = 0. \qquad (2)$$

The unknown quantity in (1), the given equation, is x, while in (2), the transformed equation, it is x_1, which was taken less by a than x. Hence, the roots (which are the values of the unknown quantity) of (2) are less by a than those of (1).

In this transformation all the coefficients except the first have been changed. Restoring in (2) the value of x_1, which is $x - a$, we have

$$A(x-a)^n + B_1(x-a)^{n-1} + C_1(x-a)^{n-2} + \cdots J_1(x-a)^2$$
$$+ K_1(x-a) + L_1 = 0,^* \qquad (3)$$

which must be identical with (1), though in different form. It is seen that if we divide the first member of (3) by $x - a$, then divide the quotient by $x - a$, and so on, the successive remainders will be L_1, K_1, J_1, $\cdots$ C_1, B_1, A, the coefficients, in inverse order, of (2), which is an equation whose roots are less by a than those of the given equation.

EXAMPLES CXXXIII

1. Find the equation whose roots are each less by 2 than those of

$$x^4 - 7x^3 + 11x^2 + 7x - 12 = 0.$$

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* *The coefficients* L_1, K_1, J_1, $\cdots$ C_1, B_1, A *are, respectively, the function, its 1st derivative,* $\dfrac{\text{2d derivative}}{2}$, $\dfrac{\text{3d derivative}}{\lfloor 3}$, $\cdots$ $\dfrac{n\text{th derivative}}{\lfloor n}$, *with* x *replaced by* a.

Substituting $x_1 + a$ for x in $f(x) = 0$ and developing by Taylor's Formula, using the form in Art. 452, we have

$$f(x) = f(x_1 + a) = f(a) + f'(a)x_1 + f''(a)\frac{x_1^2}{2} + f'''(a)\frac{x_1^3}{\lfloor 3} + f^{\text{iv}}(a)\frac{x_1^4}{\lfloor 4} + \text{etc.} = 0.$$

Hence $L_1 = f(a)$, $K_1 = f'(a)$, $J_1 = \dfrac{f''(a)}{2}$, $\cdots$ $B_1 = \dfrac{f^{n-1}(a)}{\lfloor n-1}$, $A = \dfrac{f^n(a)}{\lfloor n}$.

If x_1 is a small fraction, for an approximate value the higher powers may be neglected, and we have

$f'(a)x_1 + f(a) = 0$, whence $x_1 = -\dfrac{f(a)}{f'(a)}$, approximately. [See Art. 549.]

Solution. By the theorem the successive remainders obtained by dividing by $x - 2$, then dividing the quotient by $x - 2$, and so on, will be, in inverse order, the coefficients sought. These remainders are obtained with great facility by synthetic division, as follows:

$x^4 - 7x^3$	$+ 11x^2$	$+ 7x$	$- 12$	$\lfloor 2$
-5	1	9	6	1st rem.
-3	-5	-1	2d rem.	
-1	-7	3d rem.		
1	4th rem.			
1	5th rem.			

Hence the equation whose roots are each less by 2 than those of the given equation is

$$x_1^4 + x_1^3 - 7x_1^2 - x_1 + 6 = 0,$$

or, omitting subscripts, $x^4 + x^3 - 7x^2 - x + 6 = 0.$

Let the student find by Art. 537 the roots of each equation, and see whether those of the second are less by 2 than the corresponding ones of the first.

2. Find the equation whose roots are less by 3.14 than those of $2x^4 - 5x^3 - 3x^2 + 4x - 23 = 0$.

Solution. We may diminish the roots first by 3, then by .1 more, and then by .04 more, without writing the intermediate equations, as follows:

$2x^4 - 5x^3$	$-3x^2$	$+4x$	-23	$\lfloor 3$
1	0	4	$-11_{(1)}$	$\lfloor .1$
7	21	$67_{(1)}$	7.3192	
13	$60_{(1)}$	6.192	$-3.6808_{(2)}$	$\lfloor .04$
$19_{(1)}$	1.92	73.192	3.28970432	
.2	61.92	6.386	$-.39109568_{(3)}$	
19.2	1.94	$79.578_{(2)}$		
.2	63.86	2.664608		
19.4	1.96	82.242608		
.2	$65.82_{(2)}$	2.696544		
19.6	.7952	$84.939152_{(3)}$		
.2	66.6152			
$19.8_{(2)}$	.7984			
.08	67.4136			
19.88	.8016			
.08	$68.2152_{(3)}$			
19.96				
.08				
20.04				
.08				
$20.12_{(3)}$				

The numbers marked (**1**), together with the first coefficient, which remains unchanged, are the coefficients of the equation whose roots are less by 3 than those of the given equation.

The numbers marked (**2**), together with the first coefficient, are the coefficients of the equation whose roots are less by .1 than those of the equation whose coefficients are marked (**1**), and, consequently, less by 3.1 than those of the given equation.

The numbers marked (**3**), together with the first coefficient, are the coefficients of the equation whose roots are less by .04 than those of the equation whose coefficients are marked (**2**), less by .14 than those of the equation whose coefficients are marked (**1**), and less by 3.14 than those of the given equation. Hence the equation whose roots are less by 3.14 than those of the given equation is

$$2x^4 + 20.12x^3 + 68.2152x^2 + 84.939152x - .39109568 = 0.$$

3. Find the equation whose roots are less by 3.213 than those of $x^3 + 11x^2 - 102x + 181 = 0$.

4. Find the equation whose roots are less by 2.85 than those of $x^4 - 12x^2 + 12x - 3 = 0$.

549. Theorem. *When a root of an equation is a small fraction, it is approximately equal to the absolute term divided by the coefficient of the first power of x.*

Dem. Let x_1, a small fraction, be a root of the equation

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \cdots Kx + L = 0.$$

Substituting, we have

$$Ax_1^n + Bx_1^{n-1} + Cx_1^{n-2} + \cdots Kx_1 + L = 0.$$

Now since x_1 is a small fraction, the terms with powers above the first are so small that the equation will be little affected by neglecting them and retaining only $Kx_1 + L = 0$, which gives

$$x_1 = -\frac{L}{K}, \text{ approximately.}$$

Illustration. Consider the equation

$$9x^3 - x^2 + 9x - 1 = 0.$$

When $x = 0$, $f(x) = -1$, and when $x = 1$, $f(x) = +16$.

Hence, there is a real root between 0 and 1, and it is much nearer 0 than 1, *i.e.*, it is a small fraction. Then x^3 and x^2 are much smaller than x, and

the equation will be little affected by neglecting the terms containing them, giving $9x - 1 = 0$, whence $x = \frac{1}{9} = .11$, approximately. Indeed, in this case, the quotient, $\frac{1}{9}$, of the absolute term divided by the coefficient of the 1st power of x is the exact root.

550. The principle of Horner's Method of finding a root of an equation, exactly if commensurable, approximately if incommensurable, is this: Suppose the integral part of the root to have been found by Art. 542. If by Art. 548 we find the equation whose roots are less by this integral part than those of the given equation, the corresponding root of this transformed equation will be the decimal part of the root of the given equation. Now, by Art. 549, this part is approximately equal to the absolute term of this transformed equation divided by the coefficient of the 1st power of x in the same equation (*i.e.*, the first remainder divided by the second remainder obtained in successively dividing $f(x)$ by x minus the integral part of the root). Using only the first decimal figure of the quotient, we may find the equation whose roots are less by this decimal figure than those of the first transformed equation, and, as before, obtain another figure of the root by dividing the absolute term of this second transformed equation by the coefficient of the 1st power of x in the same equation, and so on.

551. Reverting to example 2, page 333, we may see an application of the method. By the principle of Art. 541, it is found that a real root lies between 3 and 4, *i.e.*, one root is $3 +$ a decimal fraction.

The equation whose roots are less by 3 has for its corresponding root this decimal fraction, and this fraction, by Art. 549, is approximately the first remainder divided by the second, *i.e.*, $11 \div 67 = .1$, approximately, using only the first decimal figure.

The equation whose roots are less by .1 than those of the last has for its corresponding root the remaining figures of the decimal fraction, and this is approximately the new first remainder divided by the new second remainder, *i.e.*, $3.6808 \div 79.578$ (it is sufficient to use $3.68 \div 79) = .04$, approximately.

For the third and probably the fourth figures of the decimal part of the root we have $.39109568 \div 84.939152 = .0046$. Hence the root to four decimal places is 3.1446.

For a farther application of the method, let it be required to find to four decimal places the positive root of

$$x^3 + 2x^2 - 9x - 22 = 0.$$

x	$f(x)$
0	−
1	−
2	−
3	−
4	+

As indicated in the margin, the one positive root lies between 3 and 4, *i.e.*, it is 3 + a decimal fraction. We must, therefore, diminish the roots by 3, and for a second figure of the root, divide the first remainder by the second, and so on, the work being as follows:

$x^3 + 2x^2$	$-9x$	-22	3.1273
5	6	$-4_{(1)}$	
8	$30_{(1)}$	3.111	
$11_{(1)}$	1.11	$-.889_{(2)}$	
.1	31.11	.649128	
11.1	1.12	$-.239872_{(3)}$	
.1	$32.23_{(2)}$		
11.2	.2264		
.1	32.4564		
$11.3_{(2)}$	.2268		
.02	$32.6832_{(3)}$		
11.32			
.02			
11.34			

1st figure = 3,

2d figure = 4 ÷ 30 = .1,

3d figure = .889 ÷ 32.23 = .02,

4th and 5th figures = .239872 ÷ 32.6832 = .0073

552. Carefully note the following observations:

1. As we find a root by starting with a value too small and increasing it by annexing figure after figure, a change of sign of the absolute term in any of the transformed equations, *i.e.*, the first remainder of any of the successive sets of divisions, would indicate that we had passed beyond the value of the root (Art. 541), and that the last figure must be diminished. The figure to be adopted in any case is the largest number which, in the process of diminishing the roots, will not make the sign of the absolute term different from that of the first transformed equation. If the absolute term of the first transformed equation is different from that of the given equation, it simply indicates that there is another root between 0 and the one we have started to find.

2. As the successive figures of the root are found by neglecting all but the last two terms of an equation whose roots are less by the part of the root already found than those of the given equa-

tion (Art. 549), the quotient of the absolute term of a transformed equation by the coefficient of the 1st power of x in the same equation is liable to be too large, though seldom so beyond the second figure of the root. To avoid this for the second figure of the root, we may solve, approximately, the quadratic obtained by neglecting, not all but two, but all but three, of the terms of the first transformed equation. This is also necessary if, in any case, the next to the last term is 0.

3. Since $x_1 = -\frac{L}{K}$, approximately (Art. 549), in which L is the first and K the second remainder, K and L must have opposite signs, for otherwise the quotient would be the amount to be subtracted from, instead of added to, the part of the root already found. If after the first transformation the first and second remainders have like signs, the next figure of the root cannot be found by division, but must be found by the same kind of trial which gives the first figure.

4. If two or more roots have the same initial part, the next figure of each must be found by trial, after which the process is the same as for other cases.

5. Ordinarily the fourth decimal figure will be correctly given by writing two figures of the third quotient.

6. The negative roots of an equation are found by finding the positive roots of the equation obtained by changing the signs of the odd or the even powers (Art. 526).

7. When all but one of the roots have been found, the remaining one becomes known by adding the sum of the known roots to the coefficient of the second term (the coefficient of x^{n-1}) of the given equation and changing the sign of the result (Art. 539).

EXAMPLES CXXXIV

Find to three or four decimal places the real roots of the following:

1. $x^3 - 4x^2 - 6x + 8 = 0$.

SOLUTION. As indicated in the margin, the positive roots are between 0 and 1, 4 and 5, while the negative root is between -1 and -2.

x	$f(x)$
0	+
1	−
2	−
3	−
4	−
5	+

x	$f(-x)$
0	−
1	−
2	+

To find the negative root the operation is as follows:

$x^3 + 4x^2$	$-6x$	-8	1.8004
5	1	$-9_{(1)}$	
6	$5_{(1)}$	8.992	
$7_{(1)}$	6.24	$-.008_{(2)}$	
.8	11.24		
7.8	6.88		
.8	$18.12_{(2)}$		
8.6			
.8			
$9.4_{(2)}$			

Since $9 \div 5$ cannot give the first decimal figure, we use

$$7x^2 + 5x - 9 = 0, \quad \text{(Obs. 3)}$$

whence, $x^2 + \frac{5}{7}x = \frac{9}{7}$,

and $x = -\frac{5}{14} \pm \frac{11.6}{14} = .8.$

The 2d quotient $= .008 \div 18.12 = .0004$.

Let the student show that the root between 4 and 5 is 4.8922.

The sum of these two roots and the second coefficient is $-.9082$. Hence the third root is .9082 (Obs. 7).

2. $8x^3 - 17x^2 - 16x + 34 = 0.$

Solution. By trial the roots are seen to lie between 1 and 2, 2 and 3, -1 and -2. The operation of finding the largest root is as follows:

$8x^3 - 17x^2$	$-16x$	$+34$	2.125
-1	-18	$-2_{(1)}$	
15	$12_{(1)}$	1.518	
$31_{(1)}$	3.18	$-.482_{(2)}$	
.8	15.18	.382224	
31.8	3.26	$-.099776_{(3)}$	
.8	$18.44_{(2)}$	.099776	
32.6	.6712	000000	
.8	19.1112		
$33.4_{(2)}$	.6744		
.16	$19.7856_{(3)}$		
33.56	.1696		
.16	19.9552		
33.72			
.16			
$33.88_{(3)}$			
.04			
33.92			

1st figure $= 2$,

2d figure $= 2 \div 12 = .1$,

3d figure $= .482 \div 18.44 = .02$,

4th figure $= .099776 \div 19.7856 = .005$.

As the absolute term reduces to 0, the exact root has been found.

The other two roots might also be found by Horner's Method; but since this root is commensurable, and its exact value has been found, a much better way is to depress the equation and solve the resulting quadratic, as follows:

$$\begin{array}{rrrr|l} 8x^3 & -17x^2 & -16x & +34 & 2\frac{1}{8} \\ & 0 & -16 & 0 & \end{array}$$

Hence $8x^2 - 16 = 0$, whence $x^2 = 2$, and $x = \pm\sqrt{2}$.

3. $x^3 - 6x^2 + 5x + 11 = 0$.

4. $x^3 + 10x^2 + 5x - 260 = 0$.

5. $8x^3 - 65x^2 + 140x - 33 = 0$.

6. $x^3 + 3x^2 + 4x + 5 = 0$.

7. $x^3 + x = 1000$.

8. $4x^3 - 9x^2 - 52x + 117 = 0$.

9. $x^3 - 3x^2 - 3x = -18$.

10. $x^3 - 3x^2 - 4x + 13 = 0$.

11. $x^4 + 4x^3 - 5x^2 - 18x = 22$.

12. $x^4 - 6x^3 + 3x^2 + 30x = 50$.

13. $x^4 + 4x^3 - 4x^2 - 11x + 4 = 0$.

14. $2x^4 + 5x^3 + 4x^2 + 3x = 8002$.

15. $x^5 - 4x - 2000 = 0$.

16. $x^5 - 4x^4 + 7x^3 - 863 = 0$.

17. The radius of a sphere is 9 inches, and the volume of a segment of one base cut from it is one fourth of that of the sphere. Find its altitude, the volume of a spherical segment of one base in terms of its altitude and the radius of the sphere being $\pi\left(rh^2 - \frac{h^3}{3}\right)$.

18. The weight of a sphere 1 foot in diameter is $\frac{125\pi}{32}$ pounds. When floated in a full vessel of water weighing 62.5 pounds per cubic foot, it causes an overflow of $\frac{\pi}{16}$ cubic feet of water. Find the depth to which it sinks, the law being that the weight of the water displaced equals the weight of the floating body.

19. A rectangular inner court, 100 feet long and 50 feet wide, has its principal openings at diagonally opposite corners. Between these openings matting 7 feet wide and with square ends, the corners just touching the walls, is laid (*i.e.*, the matting is a rectangle inscribed within a rectangle). Find the length of the matting.

553. The extraction of the nth root of any number, as a, is the same as finding the positive, real root of the equation,

$$x^n + 0\,x^{n-1} + 0\,x^{n-2} + \cdots - a = 0.$$

Hence Horner's Method will give to any required degree of accuracy any root of a given number. By the principle of Art. 471, the use of a table of logarithms will give the result with far less labor than will either the elementary method or Horner's Method.

EXAMPLES CXXXV

Perform by Horner's Method the following indicated operations:

1. $\sqrt[3]{29791}$.

2. $\sqrt{70444997}$.

3. $\sqrt[3]{5.6847}$.

4. $\sqrt[5]{5}$.

STURM'S THEOREM AND METHOD

554. The change of sign of $f(x)$ as we pass from one value of x to another usually reveals with little difficulty the situation of the real roots of an equation (Art. 541). In infrequent cases, however, there are difficulties, the nature of them being this: After locating, by Art. 541, all the real roots which the substitution of integral numbers will reveal, we are sometimes uncertain whether, 1st, the remaining roots are imaginary, 2d, three or higher odd number of real roots lie between two consecutive numbers that cause $f(x)$ to change sign, or, 3d, two or higher even number of real roots lie between two consecutive numbers that do not cause $f(x)$ to change sign. In the latter case the method of ex. 16, p. 327, usually removes the uncertainty and locates the roots if real; but in exceptional cases the uncertainty remains or is removed only after much labor. In 1829 Jacques Charles François Sturm (1803–1855), a Swiss mathematician, who afterward became a member of the French Academy, Professor of Mathematics in the Polytechnic School and Professor of Mechanics in the *Faculté des Sciences* in Paris, discovered a theorem by means of which, in all cases, the number and situation of the real roots of a numerical equation may be found. As its application is laborious, it is used only as a last resort when the method of Art. 541 fails or is not readily applicable.

555. If $f(x)$ and its first derivative be treated as in the process of finding the **h. c. d**, except that each remainder, before being used as the next divisor, have its signs changed and that no negative factors be introduced or rejected, these successive remainders with their signs changed constitute what are called *Sturm's Functions,* or the *Sturmian Functions.*

Thus, let $$f(x) = x^3 - 4x^2 - x + 4 = 0.$$

The first derivative is $3x^2 - 8x - 1$. Dividing $x^3 - 4x^2 - x + 4$ by $3x^2 - 8x - 1$, first multiplying the former by 3 to avoid fractions, as in the process of finding the **h. c. d.** (Art. 111), the first remainder of lower degree than the divisor is found to be $-19x + 16$. Hence $19x - 16$ is the first Sturmian Function. Similarly, the next remainder is found to be -2025. Hence 2025 is the second Sturmian Function.

556. Notation. As the first member of an equation is represented by $f(x)$ and its first derivative by $f'(x)$, we shall represent Sturm's Functions by $f_1(x)$, $f_2(x)$, $f_3(x)$, etc.

Applying this notation to the above example, we have

$$\begin{aligned} f(x) &= x^3 - 4x^2 - x + 4, \\ f'(x) &= 3x^2 - 8x - 1, \\ f_1(x) &= 19x - 16, \\ f_2(x) &= 2025. \end{aligned}$$

557. If $f(x)$ and $f'(x)$ have no common factor, the last Sturmian Function does not contain x; if they have a common factor, the last Sturmian Function is their **h. c. d.**

558. Sturm's Theorem. *If in $f(x)$, $f'(x)$, and Sturm's Functions two different numbers be substituted, the difference in the number of variations in the two cases will be the number of real roots of $f(x) = 0$ that lie between the substituted numbers, multiple roots being counted but once.*

Dem. I. When $f(x) = 0$ has no equal roots.

1st. Two consecutive functions cannot vanish, i.e., *become* 0, *for the same value of x, and, consequently, cannot change signs simultaneously.*

Let the several quotients in the process of obtaining Sturm's Functions be represented by q, q', q'', q''', etc.; then by the principles of division we have

$$f(x) = f'(x)\,q - f_1(x), \tag{1}$$
$$f'(x) = f_1(x)\,q' - f_2(x), \tag{2}$$
$$f_1(x) = f_2(x)\,q'' - f_3(x), \tag{3}$$
$$f_2(x) = f_3(x)\,q''' - f_4(x), \tag{4}$$

etc., etc., etc.

Now suppose that some value of x causes two consecutive functions, as $f_1(x)$ and $f_2(x)$, to vanish. Then from (3), $f_3(x)=0$; from (4), $f_4(x)=0$; and so on to the last function. But the last function does not contain x (Art. 557), and cannot vanish for *any* value of x. Hence two consecutive functions cannot vanish for the same value of x, and, consequently, cannot change signs simultaneously.*

2d. The changing of sign (for different values of x) of any function after first, $f(x)$, has no effect on the number of variations.

The last function cannot change sign for any value of x, as it does not contain x, and the other functions can change signs only by passing through 0. Now, when any function, as $f_2(x)$, becomes 0, (3) becomes $f_1(x)=-f_3(x)$, *i.e.*, the adjacent functions have opposite signs, and, by 1st, neither of these can change sign when $f_2(x)$ changes. Hence, if for a value of x a little less than that which makes $f_2(x)=0$ the signs of $f_1(x)$, $f_2(x)$, and $f_3(x)$, in order, are $+ + -$, then for a value a little greater the signs in order are $+ - -$, giving only one variation in either case.

3d. One variation, and only one, is lost when $f(x)$ vanishes for increasing values of x, i.e., *when x increases through a root of $f(x)=0$.*

Taking values a little less and a little greater than x and developing by Taylor's Formula, using the form of Art. 452, we have

$$f(x \mp h)=f(x) \mp f'(x)h+f''(x)\frac{h^2}{2} \mp f'''(x)\frac{h^3}{\underline{|3}}+\text{etc.}$$

Let a be a root of $f(x)=0$. Then $f(x)$ will vanish, and we shall have

$$f(a \mp h)=\mp f'(a)h+f''(a)\frac{h^2}{2} \mp f'''(a)\frac{h^3}{\underline{|3}}+\text{etc.}$$

By taking h small enough, the second member will have the same sign as its first term. Hence when x is a little smaller than

* A function can change sign only by passing through 0 or ∞. As the functions under consideration are integral functions of x, they cannot become ∞ for any finite value of x.

a root a, $f(a-h)$ and $f'(a)$ have opposite signs, and when x is a little greater than a root a, $f(a+h)$ and $f'(a)$ have the same sign; *i.e.*, there is a loss of one variation when x increases through a root of $f(x)=0$.

Now, as there is a loss of one variation whenever x increases through a root of $f(x)=0$, causing $f(x)$ to change sign, and no change in the number of variations for the change of sign of any of the other functions, the difference in the number of variations when any two numbers are substituted for x will be the number of real roots between these numbers.

II. When $f(x)=0$ has equal roots.

When $f(x)=0$ has equal roots, the **h. c. d.** *of $f(x)$ and $f'(x)$, which contains them (Art. 545), is a factor of all the functions, and, consequently, does not affect the number of variations obtained from those functions.*

Since a divisor of two quantities is a divisor of their difference (Art. 110), equation (1) shows that the **h. c. d.** of $f(x)$ and $f'(x)$ is also a divisor of $f_1(x)$; then equation (2) shows that it is a divisor of $f_2(x)$; then equation (3) shows that it is a divisor of $f_3(x)$; and so on. If the exact value of a multiple root should be substituted, the **h. c. d.** would be 0, and all the functions would vanish; but if the **h. c. d.** is + for any particular value of x, its presence in all the functions will not affect the signs, and if it is −, it will change all the signs, thus making no change in the number of variations. But since the Sturmian Functions terminate with the **h. c. d.** (Art. 557), the number of them will be less by the degree of the **h. c. d.** than it would be if there were no multiple roots and the division were continued until a numerical remainder should be reached. Now the number of times a multiple root is repeated in the **h. c. d.** is one less than in the original equation (Art. 545). Hence the multiple roots occur but once each in the other factor of $f(x)$, and the difference in the number of variations when any two numbers are substituted in the functions will be the number of real roots between these limits, each multiple root being counted but once. The **h. c. d.** will give the number and also the value of the equal roots (Art. 545).

559. Cor. *The number of variations lost as x increases from $-\infty$ to 0 is the number of negative roots, and the number lost as x increases from 0 to $+\infty$ is the number of positive roots; while the number lost as x increases from $-\infty$ to $+\infty$ is the total number of real roots.*

560. Sch. 1. When $-\infty$ or $+\infty$ is substituted, the sign of any function is the same as the resulting sign of its first term.

561. Sch. 2. As only the sign of the last function is used when there are no equal roots, the numerical value need not be found.

EXAMPLES CXXXVI

Find by Sturm's Method the number and situation of the real roots of the following:

1. $x^4 - 8x^3 + 19x^2 - 12x + 2 = 0.$

The reason for applying to this example Sturm's Method instead of the method in Art. 541 is this: All the terms of $f(-x)$ are $+$, showing that the equation has no negative roots; and evaluating $f(x)$ for positive numbers, the results are as in the margin, leaving it uncertain whether the roots are all imaginary, two real and two imaginary, or all real. We only know that if the roots are all real, two of them lie between 0 and 1, and two of them between 3 and 4, as these numbers come the nearest to satisfying the equation. Evaluating for the tenths between these numbers would, in this case, remove the uncertainty, but we could not know that beforehand.

x	$f(x)$
0	$+\ 2$
1	$+\ 2$
2	$+\ 6$
3	$+\ 2$
4	$+\ 2$
5	$+\ 42$

The application of Sturm's Method is as follows:

$$f(x) = x^4 - 8x^3 + 19x^2 - 12x + 2,$$

$$\tfrac{1}{2}f'(x) = 2x^3 - 12x^2 + 19x - 6,$$

$$f_1(x) = 5x^2 - 20x + 8,$$

$$f_2(x) = x - 2,$$

$$f_3(x) = 12.$$

x	$f(x)$	$f'(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	Variations
$-\infty$	$+$	$-$	$+$	$-$	$+$	4
0	$+$	$-$	$+$	$-$	$+$	4
$+\infty$	$+$	$+$	$+$	$+$	$+$	0

The loss of 4 variations between 0 and ∞ shows that all the roots are real and that they are all positive. Again,

x	$f(x)$	$f'(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	Variations
0	+	−	+	−	+	4
1	+	+	−	−	+	2
2	+	−	−	0	+	2
3	+	0	−	+	+	2
4	+	+	+	+	+	0

Hence there are two real roots between 0 and 1, and two real roots between 3 and 4.

Evaluating for the tenths between these limits, the roots are found to lie between .2 and .3, .5 and .6, 3.4 and 3.5, 3.7 and 3.8. As the roots are now separated, the remaining figures are found by Horner's Method.

2. $x^3+6x^2+10x-1=0.$

3. $x^4-2x^3-6x^2+10x+5=0.$

4. $x^4-2x^3-8x^2+12x+12=0.$

5. $x^4-2x^3+4x-4=0.$

6. $2x^5-15x^4+28x^3+9x^2-64x+42=0.$

7. $x^6-8x^5+18x^4-2x^3-49x^2+78x-42=0.$

8. $x^6-4x^5-11x^4+46x^3-10x^2-76x+56=0.$

9. $x^6-2x^5-5x^4+8x^3+8x^2-8x-4=0.$

SOLUTION

$$f(x)=x^6-2x^5-5x^4+8x^3+8x^2-8x-4,$$

$$\tfrac{1}{2}f'(x)=3x^5-5x^4-10x^3+12x^2+8x-4,$$

$$f_1(x)=10x^4-13x^3-30x^2+26x+20,$$

$$f_2(x)=27x^3-10x^2-54x+20,$$

$$f_3(x)=x^2-2,$$

$$f_4(x)=0.$$

As the last remainder is 0, $f(x)$ and $f'(x)$ have a common divisor, and the given equation has multiple roots (Art. 545). As the **h. c. d.**, x^2-2, occurs

in every function, its presence will make no change in the number of variations produced by these functions (II.). Hence we evaluate the functions as they stand.

x	$f(x)$	$f'(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	Variations
$-\infty$	+	−	+	−	+	4
0	−	−	+	+	−	2
$+\infty$	+	+	+	+	+	0
− 1	+	+	−	+	−	3
− 2	+	−	−	−	+	2
1	−	+	+	−	−	2
2	−	−	+	+	+	1
3	+	+	+	+	+	0

Hence the separate roots (each multiple root being counted but one) are between 0 and − 1, − 1 and − 2, 1 and 2, 2 and 3.

Placing the **h. c. d.** of $f(x)$ and $f'(x)$ equal to 0 (Art. 545), we have

$$x^2 - 2 = 0,$$

whence $$x = \pm\sqrt{2}.$$

Hence $\sqrt{2}$ and $-\sqrt{2}$ are each double roots. These are the roots that lie between 1 and 2, and − 1 and − 2.

10. $x^6 - 4x^5 + 12x^3 - 3x^2 - 8x - 1 = 0.$

11. $x^6 - 2x^5 - 4x^4 + 12x^3 - 3x^2 - 18x + 18 = 0.$

12. $x^6 - 2x^5 - 2x^4 + 12x^3 - 15x^2 - 18x + 36 = 0.$

13. $x^6 - 2x^5 - 8x^4 + 8x^3 + 20x^2 - 8x - 16 = 0.$

RECURRING OR RECIPROCAL EQUATIONS

562. **A Recurring or Reciprocal Equation** is an equation in the normal form (Art. 510) in which the coefficients equidistant from the two ends are numerically equal, the corresponding coefficients having either all like or all unlike signs.

Thus, $$x^4 - 3x^3 + 4x^2 - 3x + 1 = 0,$$

$$3x^5 - 2x^4 + 5x^3 - 5x^2 + 2x - 3 = 0,$$

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \cdots + Cx^2 + Bx + A = 0,$$

are recurring or reciprocal equations.

563. It is evident that when the corresponding coefficients of a recurring or reciprocal equation of even degree, having, therefore, an odd number of terms, have unlike signs, the coefficient of the middle term is 0; *i.e.*, the middle term is wanting.

564. Theorem. *The reciprocal of any root of a recurring equation is also a root.*

Dem. If a satisfies the equation

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \cdots + Cx^2 + Bx + A = 0,$$

$\frac{1}{a}$ will also satisfy it; for the substitution of the former gives

$$Aa^n + Ba^{n-1} + Ca^{n-2} + \cdots + Ca^2 + Ba + A = 0,$$

and the substitution of the latter gives

$$\frac{A}{a^n} + \frac{B}{a^{n-1}} + \frac{C}{a^{n-2}} + \cdots + \frac{C}{a^2} + \frac{B}{a} + A = 0,$$

which, when cleared of fractions, becomes

$$A + Ba + Ca^2 + \cdots + Ca^{n-2} + Ba^{n-1} + Aa^n = 0,$$

and this is the same as the equation obtained by substituting a.

Sch. It is on account of this reciprocal relation of the roots of recurring equations that they are called also **Reciprocal Equations.**

565. Theorem. *A recurring equation of odd degree has* $+1$ *or* -1 *as a root, according as the corresponding coefficients have unlike or like signs.*

Dem. When the corresponding terms have unlike signs, the substitution of $+1$ for x will cause them to cancel each other; and when they have like signs, the substitution of -1 for x will also cause them to cancel each other, since one of these terms is an even and the other an odd power of x.

566. Theorem. *A recurring equation of even degree, whose corresponding coefficients have unlike signs, has both* $+1$ *and* -1 *as roots.*

Dem. The equation has the form

$$x^{2n} + Ax^{2n-1} + Bx^{2n-2} + \cdots - Bx^2 - Ax - 1 = 0.$$

It is evident that both $+1$ and -1 will cause corresponding terms to cancel.

567. A recurring equation is said to be in the **Standard Form** when it is of even degree and its corresponding coefficients have like signs.

568. Theorem. *Every recurring equation not in the standard form may be reduced to that form.*

DEM. If the equation is of odd degree, it may be divided by $x-1$ or $x+1$, according as its corresponding coefficients have unlike or like signs (Arts. 565 and 515), reducing it to the standard form.

If it is of even degree, and its corresponding coefficients have unlike signs, it may be divided by $x+1$ and $x-1$, or x^2-1 (Arts. 566 and 515), reducing it to the standard form.

569. Theorem. *Any recurring equation in the standard form may be reduced to an ordinary equation of half the degree.*

DEM. The equation has the form

$$x^{2n} + Ax^{2n-1} + Bx^{2n-2} \cdots Mx^n \cdots Bx^2 + Ax + 1 = 0. \qquad (1)$$

Dividing by x^n and grouping the terms, we have

$$\left(x^n + \frac{1}{x^n}\right) + A\left(x^{n-1} + \frac{1}{x^{n-1}}\right) + B\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \cdots M = 0. \qquad (2)$$

Let $$x + \frac{1}{x} = y;$$

then $$\left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2} = y^2,$$

whence $$x^2 + \frac{1}{x^2} = y^2 - 2.$$

Similarly, $$x^3 + \frac{1}{x^3} = y^3 - 3y,$$

$$x^4 + \frac{1}{x^4} = y^4 - 4y^2 + 2,$$

and so on.

Hence $$x^n + \frac{1}{x^n} = y^n - ny^{n-2} + \cdots$$

Now, if these values be substituted in (2), there will result an equation of the nth degree, which is half that of the given equation.

EXAMPLES CXXXVII

Solve the following:

1. $x^5 - 11x^4 + 17x^3 + 17x^2 - 11x + 1 = 0.$

SOLUTION. By Art. 565, -1 is a root of this equation; hence the equation is divisible by $x+1$ (Art. 515).

$$\begin{array}{cccccc|l} x^5 & -11x^4 & +17x^3 & +17x^2 & -11x & +1 & -1 \\ & -12 & 29 & -12 & 1 & 0 & \end{array}$$

The depressed equation is

$$x^4 - 12x^3 + 29x^2 - 12x + 1 = 0$$

Reducing to an equation of half the degree by dividing by x^2 and regarding $x + \frac{1}{x}$ as the unknown quantity (Art. 569), we have

$$x^2 + \frac{1}{x^2} - 12\left(x + \frac{1}{x}\right) = -29;$$

or, adding 2 to both members,

$$\left(x + \frac{1}{x}\right)^2 - 12\left(x + \frac{1}{x}\right) = -27,$$

whence (Art. 361), $\qquad x + \frac{1}{x} = 6 \pm 3 = 9$ or 3.

From $\qquad x + \frac{1}{x} = 9$

we have $\qquad x = \frac{1}{2}(9 \pm \sqrt{77}).$

From $\qquad x + \frac{1}{x} = 3$

we have $\qquad x = \frac{1}{2}(3 \pm \sqrt{5}).$

Hence the roots are -1, $\frac{1}{2}(9 \pm \sqrt{77})$, and $\frac{1}{2}(3 \pm \sqrt{5})$.

2. $x^4 - 3x^3 + 4x^2 - 3x + 1 = 0.$
3. $x^4 - 5x^3 + 6x^2 - 5x + 1 = 0.$
4. $x^4 - x^3 + x - 1 = 0.$
5. $x^4 + 7x^3 - 7x - 1 = 0.$
6. $x^4 + 4ax^3 - 19a^2x^2 + 4a^3x + a^4 = 0.$
7. $ax^4 - 2x^3 + 2x - a = 0.$
8. $6x^5 - 11x^4 - 33x^3 + 33x^2 + 11x - 6 = 0.$
9. $3x^5 - 2x^4 + 5x^3 - 5x^2 + 2x - 3 = 0.$
10. $4x^6 - 24x^5 + 57x^4 - 73x^3 + 57x^2 - 24x + 4 = 0.$

570. **A Binomial Equation** is an equation in the form $x^n \pm a = 0$. The n roots of the equation are called the n *nth roots of* $\mp a$. For example, the solution of the equation $x^3 - 1 = 0$, or $x^3 = 1$, gives the *three cube roots of unity*.

571. **Theorem.** *Every binomial equation can be reduced to the form* $y^n \pm 1 = 0$, *which may be regarded as a recurring equation and treated accordingly.*

DEM. In the form $x^n \pm a = 0$, let $x^n = ay^n$; then the equation becomes $ay^n \pm a = 0$, whence $y^n \pm 1 = 0$.

EXAMPLES CXXXVIII

Solve the following:

1. $x^5 - 1 = 0$.

SOLUTION. By Art. 565, 1 is a root of this equation; hence the equation is divisible by $x - 1$ (Art. 515).

$$\begin{array}{cccccc|l} x^5 & +\,0\,x^4 & +\,0\,x^3 & +\,0\,x^2 & +\,0\,x & -\,1 & \underline{1} \\ & 1 & 1 & 1 & 1 & 0 & \end{array}$$

The depressed equation is

$$x^4 + x^3 + x^2 + x + 1 = 0.$$

Reducing to an equation of half the degree by dividing by x^2 and regarding $x + \frac{1}{x}$ as the unknown quantity (Art. 569), we have

$$\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) = -1,$$

or, adding 2 to both members,

$$\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) = 1,$$

whence (Art. 361),

$$x + \frac{1}{x} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{5} = \frac{1}{2}\left(-1 \pm \sqrt{5}\right).$$

Solving this equation for x, we have

$$x = \tfrac{1}{4}(-1 \pm \sqrt{5} \pm \sqrt{-10 \mp 2\sqrt{5}}).$$

These four roots and the one first obtained are the five 5th roots of unity. The 5th roots of any other number are these roots multiplied by the real 5th root of that number. For example, the five 5th roots of 32 (or the five roots of the equation $x^5 - 32 = 0$, or $x^5 = 32$) are the above roots multiplied by 2.

2. $x^3-1=0$. 3. $x^3+1=0$. 4. $x^4-1=0$.

5. $x^4+1=0$. 6. $x^5+1=0$. 7. $x^5-243=0$.

8. $x^6-1=0$. 9. $x^6+1=0$.

572. From $x^3-1=(x-1)(x^2+x+1)=0$, we obtain for the three cube roots of unity

$$1,\ -\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-3},\ -\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-3}.$$

The two imaginary roots possess the peculiarity that each is the square of the other. It is, therefore, customary to write as the three cube roots of unity 1, ω, and ω^2.

GENERAL SOLUTION OF CUBICS AND BIQUADRATICS

573. *Cardan's solution of the general cubic equation*

$$x^3+px^2+qx+r=0.$$

To transform this into an equation lacking the second power, we take

$$x=y-\frac{p}{3}, \tag{1}$$

obtaining $$\left(y-\frac{p}{3}\right)^3+p\left(y-\frac{p}{3}\right)^2+q\left(y-\frac{p}{3}\right)+r=0,$$

or $$y^3+oy^2+\left(q-\frac{p^2}{3}\right)y+\frac{2}{27}p^3-\frac{1}{3}pq+r=0,$$

which has the form $$y^3+my+n=0. \tag{2}$$

To transform this into an equation of the quadratic form (Art. 359), we take

$$y=z-\frac{m}{3z}, \tag{3}$$

obtaining $$z^6+nz^3-\frac{m^3}{27}=0,$$

which gives (Art. 360)

$$z^3=-\frac{n}{2}\pm\sqrt{\frac{n^2}{4}+\frac{m^3}{27}},$$

or $$z=\sqrt[3]{-\frac{n}{2}\pm\sqrt{\frac{n^2}{4}+\frac{m^3}{27}}}.$$

Substituting this value of z in (3), we have

$$y=\sqrt[3]{-\frac{n}{2}\pm\sqrt{\frac{n^2}{4}+\frac{m^3}{27}}}-\frac{m}{3\sqrt[3]{-\frac{n}{2}\pm\sqrt{\frac{n^2}{4}+\frac{m^3}{27}}}}.$$

Multiplying numerator and denominator of the last fraction by

$$\sqrt[3]{-\frac{n}{2}\mp\sqrt{\frac{n^2}{4}+\frac{m^3}{27}}},$$

and omitting double signs, since they give no more values than do single ones, we have

$$y=\sqrt[3]{-\frac{n}{2}+\sqrt{\frac{n^2}{4}+\frac{m^3}{72}}}+\sqrt[3]{-\frac{n}{2}-\sqrt{\frac{n^2}{4}+\frac{m^3}{27}}}.$$

574. Cor. *Cardan's formula fails when all the roots are real and unequal.*

Dem. If a is one of the roots, the other two may be found by dividing the equation by $x-a$ and solving the resulting quadratic. These roots will, therefore, be embraced in the forms $b+\sqrt{c}$ and $b-\sqrt{c}$, in which, for real roots, $\sqrt{c}$ may be either rational or surd.

Since the coefficient of y^2 is 0, we have (Art. 539)

$$-a-(b+\sqrt{c})-(b-\sqrt{c})=0,$$

whence $$a=-2\,b.$$

Now the equation whose roots are $-2\,b$, $b+\sqrt{c}$, and $b-\sqrt{c}$ is (Art. 520)

$$y^3-(3\,b^2+c)\,y+2\,(b^3-bc)=0,$$

in which $$m=-(3\,b^2+c),$$

and $$n=2\,(b^3-bc).$$

When these values are substituted in Cardan's formula, instead of obtaining real roots according to the hypothesis, we have the imaginary roots

$$y=\sqrt[3]{-\left(b^3-bc\right)+\left(b^2-\frac{c}{9}\right)\sqrt{-3\,b}}+\sqrt[3]{-\left(b^3-bc\right)-\left(b^2-\frac{c}{9}\right)\sqrt{-3\,b}}.$$

This is what is called the *Irreducible Case.*

575. SCH. Since the methods heretofore given are more expeditious for solving numerical cubics, no examples are appended.

576. *Descartes' solution of the general biquadratic equation* $x^4 + ex^3 + fx^2 + gx + h = 0$.

Transforming as in Art. 573 to remove the second term, we have an equation of the form

$$y^4 + jy^2 + ky + l = 0. \tag{1}$$

Now let us assume

$$y^4 + jy^2 + ky + l = (y^2 + my + n)(y^2 + py + q) = 0. \tag{2}$$

Developing and collecting terms,

$$y^4 + jy^2 + ky + l = y^4 + \left.\begin{matrix} m \\ p \end{matrix}\right| y^3 + \left.\begin{matrix} n \\ mp \\ q \end{matrix}\right| y^2 + \left.\begin{matrix} np \\ mq \end{matrix}\right| y + nq.$$

By Art. 441 we have

$$m + p = 0,\ n + mp + q = j,\ np + mq = k,\ nq = l,$$

from which we obtain

$$n = \frac{1}{2}\left(m^2 - \frac{k}{m} + j\right),\ q = \frac{1}{2}\left(m^2 + \frac{k}{m} + j\right).$$

Substituting these values of n and q in $nq = l$, we have

$$m^6 + 2jm^4 + (j^2 - 4l)m^2 - k^2 = 0.$$

If in this equation we take

$$m = \sqrt{m_1 - \tfrac{2}{3}j},$$

we shall have a cubic equation, from which m_1 may be found by Cardan's formula, and then m, n, p, and q from the relations above. Substituting these values in the second member of (2), and equating each of the factors with 0, we find the value of y.

CHAPTER XXIV

SERIES

577. Having treated in Chapter XIV the simpler kinds of series, and in Chapter XX of the development of functions into series by the Method of Indeterminate Coefficients, Taylor's Formula, and the Binomial Theorem, we now proceed to the consideration of series in general.

SECTION I—CONVERGENCY OF SERIES

578. We have seen (Art. 438) that when a function is developed into an infinite series, the sum of the series does not equal the function unless the series is convergent (Art. 434). It is clear, therefore, that a series cannot be used for purposes of demonstration unless it is known to be convergent. It hence often becomes necessary to determine whether or not a series is convergent. There is no universal test, but the convergency or divergency of series can usually be determined by means of the following theorems.

579. Theorem. *The convergency or divergency of a series is not affected by the addition or subtraction of a finite number of terms.*

For the sum of these terms is finite and determinate.

580. Theorem. *If a series all of whose terms are positive is convergent, it is convergent when some or all of its terms are made negative.*

For the sum is greatest when all the terms are positive.

581. Note. In what follows, therefore, it will be understood that all the terms are positive unless otherwise stated.

582. Theorem. *A series is convergent if all its terms, or all after a finite number, are less than the corresponding terms of a series that*

is known to be convergent; and divergent if all its terms, or all after a finite number, are greater than the corresponding terms of a series that is known to be divergent.

The truth of this theorem is apparent.

583. To apply this theorem it is necessary to have several standard series with which other series may be compared. The following serve for many cases.

I. *The geometrical series*

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots, \tag{1}$$

which may be written in the form

$$1+\frac{1}{2}+\frac{1}{2\cdot 2}+\frac{1}{2\cdot 2\cdot 2}+\frac{1}{2\cdot 2\cdot 2\cdot 2}+\cdots, \tag{2}$$

or any other decreasing geometrical progression, is convergent.

For proof see Art. 329.

II. *The series* $1+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m}+\cdots$ *is convergent when* $m>1$, *and divergent when* $m \leqq 1$.

DEM. 1st. When $m>1$.

We have $1=1$,

$$\frac{1}{2^m}+\frac{1}{3^m}<\frac{2}{2^m},$$

$$\frac{1}{4^m}+\frac{1}{5^m}+\frac{1}{6^m}+\frac{1}{7^m}<\frac{4}{4^m},$$

etc., etc.

By adding, we have

$$1+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m}+\frac{1}{5^m}+\frac{1}{6^m}+\frac{1}{7^m}+\text{etc.}<1+\frac{2}{2^m}+\frac{4}{4^m}+\text{etc.}$$

But this last series is a geometrical progression whose ratio is $\frac{2}{2^m}$. Hence, since $m>1$, making $\frac{2}{2^m}<1$, the series is convergent.

2d. When $m=1$, giving the series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots. \tag{3}$$

Grouping the terms thus,

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots,$$

it is seen that each group is greater than $\frac{1}{2}$. Hence the series is greater than

$$1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots,$$

and is, therefore, divergent.

3d. When $m<1$.

In this case each term after the first is greater than the corresponding term of the series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots,$$

which, as shown above, is divergent.

EXAMPLES CXXXIX

Determine whether the following series are convergent or divergent:

1. $1+\frac{1}{2}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\frac{1}{\underline{|5}}+\cdots.$

Sug. Compare with (2).

2. $1+3+\frac{3^2}{2}+\frac{3^3}{\underline{|3}}+\frac{3^4}{\underline{|4}}+\frac{3^5}{\underline{|5}}+\frac{3^6}{\underline{|6}}+\cdots.$

Solution. After the 5th term each term of this series is less than the corresponding term of the geometrical progression

$$\frac{3^5}{4\cdot 4}+\frac{3^6}{4\cdot 4\cdot 4}+\frac{3^7}{4\cdot 4\cdot 4\cdot 4}+\cdots,$$

whose ratio is $\frac{3}{4}$. Therefore, by I, the given series is convergent.

3. $1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\cdots.$

4. $2+\frac{3}{4}+\frac{4}{9}+\frac{5}{16}+\cdots\frac{n+1}{n^2}+\cdots.$

Sug. Compare with (3).

584. Theorem. *A series is convergent if, from the beginning or after a finite number of terms, the ratio of each term to the preceding term is less than some quantity which is itself less than* 1.

DEM. Let the series be

$$S = \cdots + k + l + m + n + \cdots, \tag{1}$$

$$= \cdots + k\left(1 + \frac{l}{k} + \frac{m}{k} + \frac{n}{k} + \cdots\right), \tag{2}$$

$$= \cdots + k\left(1 + \frac{l}{k} + \frac{lm}{kl} + \frac{lmn}{klm} + \cdots\right). \tag{3}$$

Now if the ratio of each term of (1) to the preceding term is less than p, we have from (3)

$$S < \cdots + k(1 + p + p^2 + p^3 + \cdots).$$

Hence if $p < 1$, we have from Art. 329

$$S < \cdots + k\frac{1}{1-p},$$

and the series is convergent.

585. Cor. *A series is convergent if, from the beginning or after a finite number of terms, the ratio of each term to the preceding term is less than* 1 *and approaches* 0 *as a limit.*

For the ratio is then always less than a quantity which is itself less than 1.

586. SCH. The series (3) of Art. 583 shows that it is not sufficient in the theorem of Art. 584 to say that the ratio of each term to the preceding term is less than 1. It is less than 1 in this series, but approaches 1 as a limit as the terms are indefinitely increased.

587. Theorem. *A series is divergent if, from the beginning or after a finite number of terms, the ratio of each term to the preceding term is equal to or greater than* 1.

For the sum of the series is the sum of an infinite number of finite terms.

588. Theorem. *A series of numerically decreasing terms which are alternately + and − is convergent.*

Dem. Let the series be

$$S = a - b + c - d + e - f + \cdots. \tag{1}$$

This may be written in the forms

$$S = (a - b) + (c - d) + (e - f) + \cdots, \tag{2}$$

and

$$S = a - (b - c) - (d - e) - (f - g) - \cdots. \tag{3}$$

From (2) $\qquad S > a - b,$

and from (3) $\qquad S < a.$

Hence the series is convergent.

EXAMPLES CXL

Determine whether the following series are convergent or divergent.

1. $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots.$

2. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$

3. The logarithmic series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} - \cdots.$$

4. $\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \frac{1}{7 \cdot 8} + \cdots.$

5. $\frac{2}{1 \cdot 2} + \frac{2^2}{2 \cdot 3} + \frac{2^3}{3 \cdot 4} + \frac{2^4}{4 \cdot 5} + \cdots \frac{2^n}{n(n+1)} + \cdots.$

Solution. The limit of the ratio of the $(n + 1)$th term to the nth term is

$$\frac{2^{n+1}}{(n+1)(n+2)} \div \frac{2^n}{n(n+1)} = \left.\frac{2n}{n+2}\right]_\infty = 2.$$

Hence (Art. 587) the series is divergent.

6. $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots + \frac{n}{2^n} + \cdots.$

7. $\frac{1}{x}+\frac{1}{1+x}+\frac{1}{2+x}+\frac{1}{3+x}+\cdots$

SUG. Compare with (3), Art. 583.

8. $\frac{1}{1+x}+\frac{1}{1+2x^2}+\frac{1}{1+3x^3}+\cdots\frac{1}{1+nx^n}+\cdots.$

9. $\frac{1}{1+x}+\frac{1}{1+2x}+\frac{1}{1+3x}+\frac{1}{1+4x}+\cdots.$

10. $1+\frac{1}{2}+\frac{1}{3^2}+\frac{1}{4^3}+\cdots\frac{1}{n^{n-1}}+\cdots.$

11. $1-\frac{x}{1+a}+\frac{x^2}{1+2a}-\frac{x^3}{1+3a}+\cdots.$

12. The Exponential Series (Napierian)

$$e^x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\frac{x^4}{\lfloor 4}+\cdots+\frac{x^n}{\lfloor n}+\cdots.$$

SOLUTION. The limit of the ratio of the $(n+1)$th term to the nth term is

$$\frac{x^{n+1}}{\lfloor n+1}\div\frac{x^n}{\lfloor n}=\frac{x}{n}\bigg]_{n=\infty}=0.$$

Hence (Art. 585) the series is convergent for all values of x.

13. The Binomial Series

$$(1+x)^m=1+mx+\frac{m(m-1)}{\lfloor 2}x^2+\cdots$$
$$+\frac{m(m-1)\cdots(m-n+2)}{\lfloor n-1}x^{n-1}+\cdots.$$

SOLUTION. The limit of the ratio of the $(n+1)$th term to the nth term is

$$\frac{m(m-1)\cdots(m-n+1)x^n}{\lfloor n}\div\frac{m(m-1)\cdots(m-n+2)x^{n-1}}{\lfloor n-1}$$
$$=\frac{m-n+1}{n}x=\left(\frac{m+1}{n}-1\right)x\bigg]_{n=\infty}=-x.$$

If, then, x be numerically less than 1, the series will be convergent (Art. 584).

It may be shown that the series is convergent when $x=1$, provided $m>-1$; also when $x=-1$, provided m is positive. See C. Smith's Treatise on Algebra, Art. 338.

14. $1 + \frac{1}{2} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \cdots.$

15. $\frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \cdots + \frac{x^n}{n(n+1)} + \cdots.$

16. The exponential series

$$a^x = 1 + (\log_e a)x + (\log_e a)^2 \frac{x^2}{\underline{|2}} + (\log_e a)^3 \frac{x^3}{\underline{|3}} + \cdots.$$

17. The series whose nth or general term is $\frac{n^2}{\underline{|n}}$.

18. The series whose nth term is $\frac{n^n}{(n+1)^{n+1}}$.

SECTION II — SCALE OF RELATION OF A RECURRING SERIES

589. **A Recurring Series** is a series in which, either from the beginning or after a finite number of terms, each term is equal to the algebraic sum of the products of a fixed number of the preceding terms, multiplied, respectively, by certain quantities which remain the same throughout the series.

590. A recurring series is of the **First, Second, Third, etc., Order,** according as each term is derived from one, two, three, etc., of the preceding terms.

Thus, by the method of indeterminate coefficients we found (page 275)

$$\frac{1 - 3x - x^2}{1 - 2x - x^2} = 1 - x - 2x^2 - 5x^3 - 12x^4 - \text{etc.}$$

In this series each term after the third is equal to $2x$ times the preceding term plus x^2 times the second preceding term. Hence it is a recurring series of the second order.

591. If a recurring series of any order be written in the form

$$u_1 + u_2 + u_3 + \cdots + u_{n-2} + u_{n-1} + v_n + u_{n+1} + u_{n+2} + \cdots,$$

we shall have by definition

$$u_n = pxu_{n-1} + qx^2u_{n-2} + rx^3u_{n-3} + \cdots,$$

whence $$u_n - pxu_{n-1} - qx^2u_{n-2} - rx^3u_{n-3} - \cdots = 0,$$

which expresses the law of the series.

In this form $$1 - px - qx^2 - rx^3 - \cdots$$

constitutes the **Scale of Relation**, or the **Scale**, and p, q, r, etc., are the **Constants of the Scale**.

Thus, in the series of Art. 590 the scale of relation is $1 - 2x - x^2$, and the constants of the scale are 2, 1.

592. To extend a series some of whose terms are given, it is sufficient to make use of the constants of the scale, since the powers of the variable may be supplied by inspection.

593. Prob. *To find the constants of the scale.*

Solution. 1st. In a series of the first order.

Let the series be

$$a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \cdots,$$

and p the constant of the scale.

Then $\quad f = pe, \quad$ whence $\quad p = \dfrac{f}{e}.$

The series is a geometrical progression.

2d. In a series of the second order.

Let p, q be the constants of the scale. Then

$$f = pe + qd,$$

$$e = pd + qc,$$

from which p and q may be found.

3d. In a series of the third order.

Let p, q, r be the constants of the scale. Then

$$f = pe + qd + rc,$$

$$e = pd + qc + rb,$$

$$d = pc + qb + ra,$$

from which p, q, and r may be found.

We may proceed in the same way for series of higher order.

When the order is not known, we may assume it of the second, third, etc., order until the right order be found. If the order be assumed too high, one or more of the constants will be 0; and if

assumed too low, the error will become apparent in applying the scale found.

594. Cor. *To find the scale we must have twice as many terms of the series as there are constants in the scale.*

EXAMPLES CXLI

Find the constants of the scale in each of the following, and extend each series one term:

1. $1 + 4x + 6x^2 + 11x^3 + 28x^4 + 63x^5 + 131x^6 + \cdots$.

SOLUTION. Assuming the series of the second order, we have

$$4p + q = 6,$$
$$6p + 4q = 11,$$

whence $p = \frac{13}{10}$ and $q = \frac{4}{5}$.

If the proper constants have been found, we shall have

$$11p + 6q = 28;$$

but $11(\frac{13}{10}) + 6(\frac{4}{5}) = 19.7$.

Hence the series is not of the second order.

Next assuming it of the third order, we have

$$6p + 4q + r = 11,$$
$$11p + 6q + 4r = 28,$$
$$28p + 11q + 6r = 63,$$

whence $p = 2,\ q = -1,\ r = 3$.

If the proper constants have been found, we shall have

$$63p + 28q + 11r = 131.$$

Now $63 \cdot 2 + 28(-1) + 11 \cdot 3 = 131$.

Hence the proper constants have been found.

To find the coefficient of the next term we have

$$131 \cdot 2 + 63(-1) + 28 \cdot 3 = 283.$$

Hence the next term is $283x^7$.

2. $1 + 6x + 12x^2 + 48x^3 + 120x^4 + \cdots$.
3. $1 + 3x + 7x^2 + 17x^3 + 41x^4 + \cdots$.
4. $1 + 9x - 15x^2 + 57x^3 - 159x^4 + \cdots$.
5. $1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + \cdots$.

6. $2 + x - 3x^2 + 2x^3 + x^4 - 3x^5 + \cdots.$

7. $3 + 5x + 7x^2 + 13x^3 + 23x^4 + 45x^5 + 87x^6 + \cdots.$

SECTION III — THE *N*TH TERM OF A SERIES

595. We have learned how to find the nth term of an arithmetical series (Art. 318), a geometrical series (Art. 326), the binomial series (Art. 457), and of a recurring series by extension to the nth term by means of the constants of the scale of relation (Art. 593). A useful method of finding the nth term of a less simple series is by the *Successive Orders of Differences.*

596. The First Order of Differences of a series is the series obtained by subtracting the 1st term of the given series from the 2d, the 2d from the 3d, the 3d from the 4th, and so on. **The Second Order of Differences** is the series obtained from the first order as the first order is obtained from the given series. **The Third, Fourth, etc., Orders** are obtained similarly.

Thus,	given series,	1, 8, 27, 64, 125, etc.,
	1st order of differences,	7, 19, 37, 61, etc.,
	2d order of differences,	12, 18, 24, etc.,
	3d order of differences,	6, 6, etc.,
	4th order of differences,	0, etc.

597. Prob. *To find the first term of any order of differences.*

Solution. Let the series be $a, b, c, d, e, \cdots$

1st order of diff.,	$b - a,$	$c - b,$	$d - c,$	$e - d, \cdots$
2d order of diff.,	$c - 2b + a,$	$d - 2c + b,$	$e - 2d + c, \cdots$	
3d order of diff.,	$d - 3c + 3b - a,$	$e - 3d + 3c - b, \cdots$		
4th order of diff.,	$e - 4d + 6c - 4b + a, \cdots$			

Denoting the first terms of the respective orders of differences by D_1, D_2, D_3, D_4, etc., we have

$$\begin{aligned} D_1 &= -a + b, \\ D_2 &= a - 2b + c, \\ D_3 &= -a + 3b - 3c + d, \\ D_4 &= a - 4b + 6c - 4d + e, \\ &\text{etc.,} \qquad \text{etc.} \end{aligned}$$

The coefficients in these terms are seen to be, numerically, those of a developed binomial by the binomial theorem. Hence, when n is even,

$$D_n = a - nb + \frac{n(n-1)}{\lfloor 2}c - \frac{n(n-1)(n-2)}{\lfloor 3}d + \cdots,$$

and when n is odd,

$$D_n = -a + nb - \frac{n(n-1)}{\lfloor 2}c + \frac{n(n-1)(n-2)}{\lfloor 3}d - \cdots.$$

598. Cor. *To find the 1st term of the nth order of differences, $n+1$ terms of the series must be given.*

This is seen by inspecting the values of D_1, D_2, D_3, D_4, etc.

EXAMPLES CXLII

Find the first term of the specified order of differences in each of the following:

1. 3d and 4th of 7, 12, 21, 36, 62, etc.

SOLUTION. For the third order we have

$$D_3 = -a + 3b - 3c + d = -7 + 3\cdot 12 - 3\cdot 21 + 36 = 2.$$

For the 4th order,

$$D_4 = a - 4b + 6c - 4d + e = 7 - 4\cdot 12 + 6\cdot 21 - 4\cdot 36 + 62 = 3.$$

In practice the shortest way is to find the successive orders by subtraction.

2. 3d of 1, 3, 6, 10, 15, etc.

3. 3d and 4th of 1, 8, 27, 64, 125, etc.

4. 3d and 5th of 1, 3, 3^2, 3^3, 3^4, 3^5, etc.

599. Prob. *To find, by the successive orders of differences, the nth term of a series.*

SOLUTION. From Art. 597 we obtain

$$\begin{aligned} b &= a + D_1, \\ c &= a + 2D_1 + D_2, \\ d &= a + 3D_1 + 3D_2 + D_3, \\ e &= a + 4D_1 + 6D_2 + 4D_3 + D_4, \\ &\text{etc.,} \qquad\qquad \text{etc.} \end{aligned}$$

It is seen that the coefficients of the nth term of the series are the coefficients of the $(n-1)$th power of a binomial. Hence, writing $n-1$ for n in the coefficients of the binomial formula (Art. 457), we have

$$n\text{th term} = a + (n-1)D_1 + \frac{(n-1)(n-2)}{\underline{|2}}D_2 + \frac{(n-1)(n-2)(n-3)}{\underline{|3}}D_3 + \cdots.$$

It is evident that the nth term of a series can be found exactly only when the terms of some order of differences are 0.

EXAMPLES CXLIII

Find the terms specified in the following:

1. 12th term of 1, 5, 15, 35, 70, 126, etc.

SOLUTION. Obtaining the successive orders of differences, we have

4, 10, 20, 35, 56, etc., whence $D_1 = 4$;

6, 10, 15, 21, etc., whence $D_2 = 6$;

4, 5, 6, etc., whence $D_3 = 4$;

1, 1, etc., whence $D_4 = 1$;

0, etc., whence $D_5 = 0$.

Substituting in the formula, we have

$$n\text{th term} = 1 + 11\cdot 4 + \frac{11\cdot 10}{2}6 + \frac{11\cdot 10\cdot 9}{2\cdot 3}4 + \frac{11\cdot 10\cdot 9\cdot 8}{2\cdot 3\cdot 4} = 1365.$$

2. 12th term of 1, 3, 6, 10, 15, 21, etc.

3. 15th term of 1, 2^2, 3^2, 4^2, etc.

4. 12th term of 1, $4x$, $6x^2$, $11x^3$, $28x^4$, $63x^5$, etc.

SECTION IV — SUMMATION OF SERIES

600. We have learned how to find the sum of n terms of an arithmetical series (Art. 319), and of a geometrical series (Art. 327), and the limit of the sum of an infinite decreasing geometrical series (Art. 329). We proceed to develop methods for finding the sum of series that are less simple, though there is no general formula for summation.

601. The sign of summation is the Greek letter *sigma*, Σ. Written before the general term of a series it signifies the sum of the series obtained by making n successively equal to 1, 2, 3, 4, etc.

Thus, $$\sum \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots.$$

602. **The Generating Function** of a series is the function which, when expanded (Art. 436), produces the series.

Thus, since $\dfrac{1-3x-x^2}{1-2x-x^2} = 1 - x - 2x^2 - 5x^3 - 12x^4 - \cdots$, $\dfrac{1-3x-x^2}{1-2x-x^2}$ is the generating function of the series $1 - x - 2x^2 - 5x^3 - 12x^4 - \cdots$.

The generating function of an infinite series is the same as the sum of the series when the series is convergent, but not when the series is divergent (Arts. 436 and 438).

603. **Prob.** *To find, by the method of decomposition, the sum of a series whose general term has the form*

$$\frac{q}{n(n+p)} \text{ or } \frac{q}{n(n+p)(n+2p)},$$

or, in general,

$$\frac{q}{n(n+p)(n+2p)\cdots(n+rp)}.$$

Solution. By Art. 447, we have

$$\frac{q}{n(n+p)} = \frac{A}{n} + \frac{B}{n+p},$$

whence $$A = \frac{1}{p} \text{ and } B = -\frac{1}{p}.$$

Hence $$\frac{q}{n(n+p)} = \frac{1}{p}\left(\frac{q}{n} - \frac{q}{n+p}\right),$$

and $$\sum \frac{q}{n(n+p)} = \sum \frac{1}{p}\left(\frac{q}{n} - \frac{q}{n+p}\right) = \frac{1}{p}\left(\sum \frac{q}{n} - \sum \frac{q}{n+p}\right). \quad (1)$$

Now, when n is taken successively equal to 1, 2, 3, 4, etc., all the terms of $\sum \frac{q}{n}$ and $\sum \frac{q}{n+p}$, except the first p terms of the

former and the last p terms of the latter, will cancel, and the sum will be $\frac{1}{p}$th the sum of these remaining terms. Hence

$$\sum\frac{q}{n(n+p)}$$
$$=\frac{1}{p}\left(\text{1st } p \text{ terms of } \sum\frac{q}{n} - \text{last } p \text{ terms of } \sum\frac{q}{n+p}\right). \quad (2)$$

The sum of an infinite number of terms is $\frac{1}{p}$th the sum of the first p terms of $\sum\frac{q}{n}$, since $\left.\frac{q}{n(n+p)}\right]_\infty = 0$.

Similarly, we have

$$\sum\frac{q}{n(n+p)(n+2p)}=\frac{1}{2p}\left(\sum\frac{q}{n(n+p)}-\sum\frac{q}{(n+p)(n+2p)}\right); \quad (3)$$

and, in general,

$$\sum\frac{q}{n(n+p)(n+2p)\cdots(n+rp)}=\frac{1}{rp}\left(\sum\frac{q}{n(n+p)\cdots[n+(r-1)p]}\right.$$
$$\left.-\sum\frac{q}{(n+p)(n+2p)\cdots(n+rp)}\right). \quad (4)$$

EXAMPLES CXLIV

Find, by the method of decomposition, the sum of the following to n terms and to ∞.

1. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots\frac{1}{n(n+1)}+\cdots.$

Solution. Since $q = 1$ and $p = 1$, we have from (2)

$$\sum\frac{q}{n(n+p)} = \text{1st term of } \sum\frac{1}{n} - n\text{th term of } \sum\frac{1}{n+1}.$$

Hence
$$S_n=\frac{1}{1}-\frac{1}{n+1}=\frac{n}{n+1},$$

and
$$S_\infty = \text{1st term of } \sum\frac{1}{n} = 1.$$

The value of S_∞ is also obtained from the value of S_n by making $n=\infty$, giving

$$S_\infty = \left.\frac{n}{n+1}\right]_\infty = 1.$$

Without the use of the formula we may proceed thus:

$$\frac{1}{1\cdot 2}=\frac{1}{1}-\frac{1}{2},\ \frac{1}{2\cdot 3}=\frac{1}{2}-\frac{1}{3},\ \frac{1}{3\cdot 4}=\frac{1}{3}-\frac{1}{4},\ \text{etc.}$$

Hence the series becomes

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\frac{1}{n+1}=\frac{n}{n+1}.$$

2. $\dfrac{1}{1\cdot 3}+\dfrac{1}{2\cdot 4}+\dfrac{1}{3\cdot 5}+\cdots.$

Solution. Since $q=1$ and $p=2$, we have from (2)

$$\sum\frac{q}{n(n+p)}=\frac{1}{2}\left(\text{1st 2 terms of }\sum\frac{1}{n}-\text{last 2 terms of }\sum\frac{1}{n+2}\right).$$

Hence $\quad S_n=\dfrac{1}{2}\left(\dfrac{1}{1}+\dfrac{1}{2}-\dfrac{1}{n}-\dfrac{1}{n+2}\right)=\dfrac{1}{2}\left(\dfrac{3}{2}-\dfrac{1}{n}-\dfrac{1}{n+2}\right),$

and $\quad S_\infty=\dfrac{1}{2}\left(\dfrac{3}{2}-\dfrac{1}{n}-\dfrac{1}{n+2}\right)\Big]_\infty=\dfrac{3}{4}.$

3. $\dfrac{1}{1\cdot 4}+\dfrac{1}{2\cdot 5}+\dfrac{1}{3\cdot 6}+\dfrac{1}{4\cdot 7}+\cdots+\dfrac{1}{n(n+3)}+\cdots.$

4. $\dfrac{1}{3\cdot 8}+\dfrac{1}{6\cdot 12}+\dfrac{1}{9\cdot 16}+\cdots.$

Sug. Write in the form

$$\frac{1}{12}\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots\right),$$

and compare with ex. 1.

5. $\dfrac{1}{1\cdot 4}+\dfrac{1}{2\cdot 6}+\dfrac{1}{3\cdot 8}+\dfrac{1}{4\cdot 10}+\cdots.$

6. $\dfrac{1}{1\cdot 3}+\dfrac{1}{3\cdot 5}+\dfrac{1}{5\cdot 7}+\cdots+\dfrac{1}{(2n-1)(2n+1)}+\cdots.$

Solution. As (2) is not applicable, the denominator not having the form $n(n+p)$, we have from (1),

$$\sum\frac{q}{(2n-1)(2n+1)}=\frac{1}{2}\left(\sum\frac{1}{2n-1}-\sum\frac{1}{2n+1}\right)$$
$$=\frac{1}{2}\left(\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\cdots\frac{1}{2n-1}-\frac{1}{3}-\frac{1}{5}-\cdots-\frac{1}{2n-1}-\frac{1}{2n+1}\right).$$

Hence $$S_n = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{n}{2n+1},$$

and $$S_\infty = \frac{n}{2n+1}\Big]_\infty = \frac{1}{2}.$$

7. $\dfrac{2}{3\cdot 5} + \dfrac{2}{5\cdot 7} + \dfrac{2}{7\cdot 9} + \cdots + \dfrac{2}{(2n+1)(2n+3)} + \cdots.$

8. $\dfrac{1}{\underline{|2}} + \dfrac{2}{\underline{|3}} + \dfrac{3}{\underline{|4}} + \cdots + \dfrac{n}{\underline{|n+1}} + \cdots.$

Sug. $\dfrac{n}{\underline{|n+1}} = \dfrac{1}{\underline{|n}} - \dfrac{1}{\underline{|n+1}}.$

9. $\dfrac{2}{3\cdot 5} - \dfrac{3}{5\cdot 7} + \dfrac{4}{7\cdot 9} - \dfrac{5}{9\cdot 11} + \cdots \pm \dfrac{n+1}{(2n+1)(2n+3)} \mp \cdots.$

Solution. Since $q = n + 1$, a variable, and $p = 2$, we have from (1),

$$\sum \frac{n+1}{(2n+1)(2n+3)} = \frac{1}{2}\left(\sum \frac{n+1}{2n+1} - \sum \frac{n+1}{2n+3}\right)$$

$$= \frac{1}{2}\left(\frac{2}{3} - \frac{3}{5} + \frac{4}{7} - \cdots \mp \frac{n+1}{2n+1} - \frac{2}{5} + \frac{3}{7} - \cdots \mp \frac{n}{2n+1} \pm \frac{n+1}{2n+3}\right)$$

$$= \frac{1}{2}\left(\frac{2}{3} - 1 + 1 - \cdots \mp 1 \pm \frac{n+1}{2n+3}\right).$$

Hence, when n is even,

$$S_n = \frac{1}{2}\left(\frac{2}{3} - 1 + \frac{n+1}{2n+3}\right) = \frac{1}{2}\left(-\frac{1}{3} + \frac{n+1}{2n+3}\right);$$

and when n is odd,

$$S_n = \frac{1}{2}\left(\frac{2}{3} - \frac{n+1}{2n+3}\right).$$

$$S_\infty = \tfrac{1}{12}.$$

10. $\dfrac{1}{1\cdot 2\cdot 3} + \dfrac{1}{2\cdot 3\cdot 4} + \dfrac{1}{3\cdot 4\cdot 5} + \cdots + \dfrac{1}{n(n+1)(n+2)} + \cdots.$

Sug. Use (3).

11. $\dfrac{1}{1\cdot 3\cdot 5} + \dfrac{1}{3\cdot 5\cdot 7} + \dfrac{1}{5\cdot 7\cdot 9} + \cdots.$

12. $\dfrac{4}{1\cdot 2\cdot 3} + \dfrac{5}{2\cdot 3\cdot 4} + \dfrac{6}{3\cdot 4\cdot 5} + \cdots.$

SOLUTION. Since $q = n + 3$ and $p = 1$, we have from (3)

$$\sum\frac{q}{n(n+p)(n+2p)}=\frac{1}{2}\left(\sum\frac{n+3}{n(n+1)}-\sum\frac{n+3}{(n+1)(n+2)}\right)$$

$$=\frac{1}{2}\left(\frac{4}{1\cdot 2}+\frac{5}{2\cdot 3}+\frac{6}{3\cdot 4}+\cdots+\frac{n+3}{n(n+1)}\right.$$

$$\left.-\frac{4}{2\cdot 3}-\frac{5}{3\cdot 4}-\cdots-\frac{n+2}{n(n+1)}-\frac{n+3}{(n+1)(n+2)}\right)$$

$$=\frac{1}{2}\left(2+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}-\frac{n+3}{(n+1)(n+2)}\right).$$

Hence, by ex. 1, $S_n=\frac{1}{2}\left(2+\frac{1}{2}-\frac{1}{n+1}-\frac{n+3}{(n+1)(n+2)}\right)$

$$=\frac{1}{2}\left(\frac{5}{2}-\frac{2n+5}{(n+1)(n+2)}\right),$$

and $S_\infty=\frac{5}{4}$.

13. $\frac{1}{1\cdot 2\cdot 3}+\frac{3}{2\cdot 3\cdot 4}+\frac{5}{3\cdot 4\cdot 5}+\frac{7}{4\cdot 5\cdot 6}+\cdots$.

14. $\frac{1}{3\cdot 4\cdot 5}+\frac{2}{4\cdot 5\cdot 6}+\frac{3}{5\cdot 6\cdot 7}+\cdots$.

15. $\frac{1}{1\cdot 3}+\frac{2}{1\cdot 3\cdot 5}+\frac{3}{1\cdot 3\cdot 5\cdot 7}+\cdots\frac{n}{1\cdot 3\cdot 5\cdots(2n+1)}+\cdots$.

604. Prob. *To find, by the scale of relation, the sum of n terms of a recurring series.*

SOLUTION. Let the series be $a+bx+cx^2+dx^3+\cdots$,

whose scale of relation (Art. 591) is $1-px-qx^2$.

Though this assumes that the series is of the second order, the method is general.

Representing by S_n the sum of n terms, we have

$$S_n=a+bx+cx^2+dx^3+\cdots+lx^{n-1}.$$

Multiplying both members by the scale of relation and arranging according to the powers of x, we have

$$\begin{array}{l|l|l|l|l|l|l}
(1-px-qx^2)S_n=a+\;\; b & x+\;\; c & x^2+\;\; d & x^3+\cdots+\;\; l & x^{n-1} & & \\
\qquad\qquad -pa & \;\;-pb & \;\;-pc & \qquad -pk & \;\;-pl & x^n & \\
 & \;\;-qa & \;\;-qb & \qquad -qj & \;\;-qk & & -qlx^{n+1}.
\end{array}$$

Since, by Art. 591, the coefficients of x^2, x^3, $\cdots x^{n-1}$ are 0, we have

$$(1-px-qx^2)\,S_n = a + \begin{array}{c|c|c} b & x - pl & x^n \\ -pa & -qk & \quad -qlx^{n+1}. \end{array}$$

Hence $\quad S_n = \dfrac{a+(b-pa)\,x-(pl+qk)\,x^n - qlx^{n+1}}{1-px-qx^2}.$

605. Cor. *The sum of an infinite convergent recurring series of the second order is*

$$S_\infty = \frac{a+(b-pa)x}{1-px-qx^2}.$$

This is because the last two terms of the expression for S_n approach 0 as a limit.

606. Sch. The expression

$$\frac{a+(b-pa)x}{1-px-qx^2}$$

is the *generating function* (Art. 602), and the development of it will reproduce the original series.

EXAMPLES CXLV

Find, by the scale of relation, the generating function of each of the following:

1. $1+2x+8x^2+28x^3+100x^4+\cdots.$

Solution. We must first find the constants of the scale. From Art. 593 we have

$$8 = 2p+q,$$

$$28 = 8p+2q,$$

whence $\quad p=3, \qquad q=2.$

Since $\quad 28p+8q=100,$

the proper constants have been found.

Substituting in $\quad S = \dfrac{a+(b-pa)x}{1-px-qx^2},$

we have $\quad S = \dfrac{1-x}{1-3x-2x^2}.$

2. $1 + 9x - 15x^2 + 57x^3 - 159x^4 + \cdots$.
3. $1 + 2x + 3x^2 + 5x^3 + 100x^4 + \cdots$.
4. $1 + 5x + 9x^2 + 13x^3 + \cdots$.
5. $1 + 3x + 8x^2 + 22x^3 + 60x^4 + \cdots$.
6. $2 - 5x + 17x^2 - 65x^3 + 257x^4 - \cdots$.
7. $1 + 3x + 5x^2 + 7x^3 + \cdots$.
8. $3 + 5x + 7x^2 + 13x^3 + 23x^4 + 45x^5 + \cdots$.
9. $1 + 3x - x^2 - 5x^3 - 7x^4 - x^5 + 11x^6 + \cdots$.
10. $1 - 3x + 5x^2 + 5x^3 + 13x^4 + 61x^5 + 181x^6 + \cdots$.

607. Prob. *To find, by the method of differences, the sum of n terms of a series.*

Solution. Let the given series be

$$a,\ b,\ c,\ d,\ e,\ f, \text{ etc.,} \qquad (1)$$

and let S represent the sum of n terms.

Then S is the $(n+1)$th term of the series

$$0,\ a,\ a+b,\ a+b+c,\ a+b+c+d, \text{ etc.} \qquad (2)$$

Series (1) is the same as the first order of differences (Art. 596) of series (2), the first order of differences of (1) is the same as the second order of differences of (2), and so on.

Substituting in the formula for the nth term (Art. 599) $n+1$ for n, 0 for a, a for D_1, D_1 for D_2, etc., we have

$$S = na + \frac{n(n-1)}{2}D_1 + \frac{n(n-1)(n-2)}{\lfloor 3}D_2 + \text{etc.}$$

This formula is applicable only when the series is such that all the terms of some order of differences become 0.

EXAMPLES CXLVI

Find, by the method of differences, the sum of the following:

1. 1, 8, 21, 40, 65, etc., to 12 terms.

Solution. Obtaining the successive orders of differences, we have

$$\begin{array}{c} 1,\ 8,\ 21,\ 40,\ 65,\ \cdots \\ 7,\ 13,\ 19,\ 25,\ \cdots \\ 6,\ 6,\ 6,\ \cdots \\ 0,\ 0,\ \cdots \end{array}$$

Substituting in the formula,

$$S = 12 \times 1 + \frac{12(12-1)}{2}7 + \frac{12(12-1)(12-2)}{\underline{|3}}6 = 1794.$$

2. 1, 3, 5, 7, etc., to 20 terms.

3. 4, 14, 30, 52, 80, etc., to 13 terms.

4. 1, 2, 3, 4, 5, etc., to 50 terms.

5. 1, 5, 15, 35, 70, 126, etc., to 30 terms.

6. 7, 14, 19, 22, 23, etc., to 9 terms.

7. The following, obtained as above, are useful in Physics:

$$\Sigma n = 1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2},$$

$$\Sigma n^2 = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6},$$

$$\Sigma n^3 = 1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} = (\Sigma n)^2,$$

$$\Sigma n^4 = 1^4 + 2^4 + 3^4 + 4^4 + \cdots + n^4 = \frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30}$$

If n is a large number and only approximate results are sought, all the lower powers may be omitted, and we have

$$\Sigma n = \frac{n^2}{2}, \ \Sigma n^2 = \frac{n^3}{3}, \ \Sigma n^3 = \frac{n^4}{4}, \ \Sigma n^4 = \frac{n^5}{5},$$

and, in general,

$$\Sigma n^m = \frac{n^{m+1}}{m+1}.$$

8. $$\Sigma \frac{n(n+1)}{2} = \frac{1 \cdot 2}{2} + \frac{2 \cdot 3}{2} + \frac{3 \cdot 4}{2} + \frac{4 \cdot 5}{2} + \cdots \frac{n(n+1)}{2}$$

$$= \tfrac{1}{2}[\Sigma(n^2 + n)] = \tfrac{1}{2}(\Sigma n^2 + \Sigma n) = \frac{n(n+1)(n+2)}{6}.$$

PILES OF SPHERICAL SHOT

608. Prob. *To find the number of balls in a complete pyramid or wedge.*

Solution. *1st. A triangular pyramid.*

Let n be the number of balls in one side of the bottom course. This will also be the number of courses.

The number of balls in the successive courses, beginning at the top, is

$$1,\ 3,\ 6,\ 10,\ 15,\ \text{etc.}$$

Obtaining the successive orders of differences and substituting in the formula of Art. 607, we have

$$S = \frac{n(n+1)(n+2)}{\lfloor 3}.$$

2d. A square pyramid.

The number of balls in the successive courses is

$$1^2,\ 2^2,\ 3^2,\ 4^2,\ 5^2,\ \cdots n^2.$$

As before, we obtain

$$S = \frac{n(n+1)(2n+1)}{\lfloor 3}.$$

3d. A wedge with rectangular base and single row at top.

Let m' be the number in the top row. Then the next course, being longer by 1, will contain $2(m'+1)$; the next, $3(m'+2)$; the next, $4(m'+3)$, and so on, giving the series

$$m',\ 2m'+2,\ 3m'+6,\ 4m'+12,\ \cdots.$$

As before, we obtain

$$S = \frac{n(n+1)(3m'+2n-2)}{6}.$$

If we let m be the number of balls in the length of the base, we have

$$m' = m - n + 1,$$

and the last formula becomes

$$S = \frac{n(n+1)(3m-n+1)}{6}.$$

609. Sch. The number of balls in an incomplete pile is the number in a complete pile having the same base, diminished by the number in a complete pile whose base would be the next course above the top course of the incomplete pile.

EXAMPLES CXLVII

Find the number of balls in each of the following:

1. A triangular pile of 15 courses.
2. A triangular pile of 20 courses.

3. An incomplete triangular pile of 15 courses, having 21 balls in the top course.

4. An incomplete triangular pile whose bottom course has 15 balls on a side, and whose top course contains 28 balls.

5. A square pile of 15 courses.

6. An incomplete square pile whose bottom course has 20 balls on a side, and top course 8 on a side.

7. A rectangular pile whose bottom course is 42 balls by 20.

8. A rectangular pile whose top row contains 23 balls.

9. An incomplete rectangular pile whose top course is 12 balls by 20, and whose bottom course is 52 balls in length.

INTERPOLATION

610. Interpolation is the process of introducing between the terms of a series other terms which conform to the law of the series.

The most extensive use of interpolation is in finding intermediate terms between those given in mathematical tables, and in finding right ascensions, declinations, etc., for other times than those given in the Nautical Almanac.

611. The Argument is the variable quantity on the value of which the magnitude of the function depends.

Thus, in finding from the table on pages 300 and 301 the logarithms of given numbers, the given numbers constitute the argument. In finding from the Nautical Almanac the moon's declinations for given times, the given times constitute the argument.

When the changes of the argument are proportional to the changes of the function, no formula is needed for interpolating.

612. Prob. *To interpolate between two consecutive terms of a series a term that shall conform to the law of the series.*

Solution. Let p be the distance, in intervals, of the required term t from the first term a. Then p is an improper fraction and

is one less than the number of terms. Substituting $p+1$ for n in the formula for the nth term of a series (Art. 599), we have

$$= a + pD_1 + \frac{p(p-1)}{\underline{|2}} D_2 + \frac{p(p-1)(p-2)}{\underline{|3}} D_3 + \cdots.$$

613. The value of t is more closely approximated as more orders of differences are used, and is absolutely correct when the terms of the next order of differences are 0.

For facilitating computation the values of the coefficients of D_2, D_3, D_4, and D_5, have been computed for every hundredth part of an interval and arranged in a table. See Loomis's Practical Astronomy, page 393.

614. The formula of Art. 612 is for increasing values of the argument. A similar formula could be deduced for decreasing values of the argument. When one of these formulas gives a positive error, the other usually gives a negative error. Bessel's Formula is obtained by taking the half sum of these two formulas. Representing the argument by A, and the function by F, we have

Argument	Function	1st diff.	2d diff.	3d diff.	4th diff.	5th diff.
$A-2$	F					
		D_1				
$A-1$	F'		D_2			
		D_1'		D_3		
A	F''		D_2'		D_4	
		D_1''		D_3'		D_5
$A+1$	F'''		D_2''		D_4'	
		D_1'''		D_3''		
$A+2$	F^{iv}		D_2'''			
		D_1^{iv}				
$A+3$	F^{v}					

For interpolating between A and $A+1$, F'' is taken as the first term, making p a proper fraction; and the formula becomes

$$t = F'' + pD_1'' + \frac{p(p-1)}{\underline{|2}}\left(\frac{D_2' + D_2''}{2}\right) + \frac{p(p-1)(p-\frac{1}{2})}{\underline{|3}} D_3'$$
$$+ \frac{p(p+1)(p-1)(p-2)}{\underline{|4}}\left(\frac{D_4 + D_4'}{2}\right)$$
$$+ \frac{p(p+1)(p-1)(p-2)(p-\frac{1}{2})}{\underline{|5}} D_5 + \cdots.$$

Bessel's table (see Loomis's Practical Astronomy, page 392) gives the coefficients as far as the fifth differences for every hundredth part of an interval.

For most purposes the first three terms of the above formula, employing only first and second differences, are sufficient.*

EXAMPLES CXLVIII

1. An eclipse of the moon occurs only at the time of opposition, *i.e.*, when the sun and the moon are on opposite sides of the earth. For the eclipse of June 12, 1900, the Greenwich mean time of opposition, as given by the Nautical Almanac, is 15 h. 31 m. 30 s., and the right ascensions for the 14th, 15th, 16th, and 17th hours are as given below. Find the right ascension of the moon at the time of opposition.

SOLUTION

Argument	Function	1st diff.	2d diff.	3d diff.
hr.	hr. min. sec.	min. sec.	sec.	
June 12, 14	17 19 56.00			
		2 24.94		
15	17 22 20.94		.10	
		2 25.04		0
16	17 24 45.98		.10	
		2 25.14		
17	17 27 11.12			

In this case $F'' = 17$ h. 22 m. 20.94 s., $p = 31$ m. 30 s. $= .525$ h., $D_1'' = 2$ m. 25.04 s. $= 145.04$ s., $D_2' = .1$ s., and $D_2'' = .1$ s. Substituting in the formula of Art. 614, we have.

$$t = 7\text{ h. } 22\text{ m. } 20.94\text{ s.} + .525 \times 145.04\text{ s.} + .525\,(-.475) \times .1\text{ s.}$$
$$= 17\text{ h. } 22\text{ m. } 37.074\text{ s.}$$

2. The Greenwich mean time of conjunction at the time of the total eclipse of the sun on May 28, 1900, as given by the Nautical Almanac, is 2 hours 57 minutes 2.7 seconds, and the right ascensions of the sun for two noons before and two after are 4 hours

* For a fuller discussion of the subject of interpolation, see Loomis's Practical Astronomy, Chauvenet's Practical Astronomy, Doolittle's Practical Astronomy.

15 minutes 13.84 seconds, 4 hours 19 minutes 17.40 seconds, 4 hours 23 minutes 21.43 seconds, and 4 hours 27 minutes 25.90 seconds, respectively. Find the mean time of that phase of the eclipse which occurs at the instant of conjunction, for New Orleans, which is 6 hours west of Greenwich.

3. The right ascensions of Jupiter on four successive days at noon being 10 hours 5 minutes 38.6 seconds, 10 hours 6 minutes 18.86 seconds, 10 hours 6 minutes 59.41 seconds, and 10 hours 7 minutes 40.24 seconds, respectively, find the right ascension for midnight of the second day.

4. The cube roots of 60, 62, 64, and 66 being 3.91487, 3.95789, 4, and 4.04124, respectively, find the cube root of 63.

5. On Oct. 29, 1900, the altitude of the sun when on the meridian at Minneapolis was 31° 32′ 17.″17. The south declinations of the sun at Greenwich apparent noon on Oct. 28th, 29th, 30th, and 31st were 13° 3′ 49.″8, 13° 23′ 53.″2, 13° 43′ 43.″9, and 14° 3′ 21.″4, respectively. Find the latitude of Minneapolis, which is 6 hours 12 minutes 56.8 seconds west of Greenwich, the latitude being the complement of the sum of the meridian altitude and the south declination.

CHAPTER XXV

PERMUTATIONS AND COMBINATIONS

SECTION I—PERMUTATIONS

615. **Permutations** are the different orders in which things, taken the same number at a time, can be arranged.

Thus, the permutations of the letters a, b, c, taken two at a time, are ab, ba, ac, ca, bc, cb, and taken three at a time, abc, acb, bca, bac, cab, cba.

616. **Theorem.** *The number of permutations of n things taken r at a time is*

$$_nP_r = n(n-1)(n-2)(n-3)\cdots(n-r+1).$$

DEM. Let $_nP_2$, $_nP_3$, $_nP_4$, $\cdots$ $_nP_r$ be the number of permutations of the n things, according as 2, 3, 4, $\cdots$ r things are taken at a time.

Taken two at a time, each of the n things may be placed in turn before each of the remaining $n-1$ things, giving

$$_nP_2 = n(n-1).$$

Taken three at a time, each of the n things may be placed in turn before each of the $(n-1)(n-2)$ permutations that may be formed of the $n-1$ remaining things taken two at a time, giving

$$_nP_3 = n(n-1)(n-2).$$

Taken four at a time, each of the n things may be placed in turn before each of the $(n-1)(n-2)(n-3)$ permutations that may be formed of the $n-1$ remaining things taken three at a time, giving

$$_nP_4 = n(n-1)(n-2)(n-3).$$

The number subtracted from n in the last factor is seen to be in each case 1 less than the number of things in each permutation; hence

$$_nP_r = n(n-1)(n-2)(n-3)\cdots(n-r+1).$$

617. Cor. *The number of permutations of n things taken all at a time is* ${}_nP_n = \lfloor\underline{n}$.

In this case $r = n$.

618. Theorem. *The number of permutations of n things taken all at a time, when p of one kind are alike, q of another kind alike, and so on, is*

$$N = \frac{\lfloor\underline{n}}{\lfloor\underline{p} \times \lfloor\underline{q} \times \text{etc.}}.$$

Dem. Let N be the number of permutations that can be formed.

If in any one of these N permutations the p like things were replaced by p unlike things (unlike one another and unlike the other $n-p$ things), by changing the order of these p unlike things, leaving the other $n-p$ things unchanged, this single permutation would furnish $\lfloor\underline{p}$ permutations (Art. 617). The same change in each of the N permutations would furnish $N \times \lfloor\underline{p}$ permutations.

As the same reasoning applies to the q like things and to the other sets of like things, the replacing of the different sets of like things by unlike things would furnish $N \times \lfloor\underline{p} \times \lfloor\underline{q} \times$ etc. permutations. But the n unlike things would furnish $\lfloor\underline{n}$ permutations (Art. 617); hence

$$N \times \lfloor\underline{p} \times \lfloor\underline{q} \times \text{etc.} = \lfloor\underline{n},$$

whence
$$N = \frac{\lfloor\underline{n}}{\lfloor\underline{p} \times \lfloor\underline{q} \times \text{etc.}}.$$

619. Theorem. *If p specified things are required to occupy specified places, the number of permutations is the same as of* $n-p$ *things taken* $r-p$ *at a time.*

Dem. The only permutations possible are those arising from changes of the $n-p$ things, of which $r-p$ are in each permutation.

620. Cor. *If the p things can be rearranged among themselves, the number of permutations is* $\lfloor\underline{p}$ *times the number of permutations of* $n-p$ *things taken* $r-p$ *at a time.*

621. Theorem. *The number of permutations of n things taken all at a time in a circle is* $\lfloor\underline{n-1}$.

DEM. Since for every arrangement the things may all be shifted the same number of places in either direction around the circle, only relative positions, and not actual positions, are to be considered. Hence, if any one of the things remain in any one of the positions, all possible permutations will be formed by permuting the remaining $n-1$ things among the remaining $n-1$ positions, giving $\underline{|n-1}$ permutations (Art. 617).

622. Cor. *If each order of arrangement is limited to one direction around the circle, the number of permutations is* $\frac{1}{2}\underline{|n-1}$.

It is to be noted that a right-hand arrangement as viewed from one side of the circle is identical with the corresponding left-hand arrangement as viewed from the other side of the circle.

EXAMPLES CXLIX

1. In how many different orders can 3 hats be hung on 8 hooks?

2. How many different numbers of 3 figures each can be formed from the digits 1, 2, 3, 4, 5?

3. How many different numbers of 2 figures each can be formed from the 9 digits? How many of 3 figures each? Of 4 figures each? Of 5 figures each? Of 6 figures each? Of 7 figures each? Of 8 figures each? Of 9 figures each?

4. In how many different orders can 6 persons be seated at a dinner table?

5. In how many different orders can 3 persons occupy 7 fixed seats?

6. In how many different orders can 4 players use 6 billiard cues?

7. In how many different ways can 4 gentlemen select from 7 ladies partners for the waltz?

8. In how many different orders can a single platoon of 8 soldiers be arranged in line?

9. How many numbers between 1000 and 10000 can be formed by use of the digits 1, 2, 3, 4, 5, 6?

10. In how many different orders can the letters of the word *Algebra* be arranged? In how many the letters of the word *Mathematics?*

11. How many different signals may be made with 4 different colors, taken any number at a time?

SUG. The number $= {}_4P_1 + {}_4P_2 + {}_4P_3 + {}_4P_4$.

12. In how many different orders can 8 boys, any number at a time, enter a room?

13. In how many different orders can 12 members of a minstrel troupe arrange themselves in line on the stage, the same two always acting as end men?

See Arts. 619 and 620.

14. A shelf contains 20 books, of which 4 are single volumes and the others are in sets of 8, 5, and 3 volumes respectively. Find the number of ways in which the books can be arranged on the shelf, (*a*) when the volumes of each set remain in the order of their number, (*b*) when the volumes of each set are together, but in any order.

15. Find the number of permutations of the factors of $a^2b^3c^4$.

16. In how many different orders can a football eleven play, if the full-back, the half-back, the quarter-back, and the center-rush always play in the same positions?

17. Four of the crew of an eight-oared boat have trained to row only on the stroke side and four to row only on the bow side. In how many ways can the captain arrange his crew, (*a*) when the stroke (the rower who sets the stroke) is any one of the four on the stroke side, (*b*) when the stroke is always the same man?

18. Either A or B of a baseball nine must pitch, either C or D must catch, while E, F, and G must play on the bases. In how many ways can the captain play his team?

19. In how many different orders can 6 persons be seated at a round table?

20. In how many different orders can 7 children stand in a ring?

21. In how many different orders can a host and 7 guests sit at a round table, the host always having the guest highest in rank on his right, and the guest next in rank on his left?

22. In how many different orders can a party of 5 ladies and 5 gentlemen sit at a round table, the ladies and gentlemen sitting at alternate places?

Explain why this is $\lfloor 4 \times \lfloor 5$.

23. In how many different orders can 10 beads be strung for a rosary ring?

See Art. 622.

24. In how many different orders can 5 like pearls, 6 like rubies, and 7 like diamonds be strung for a bracelet?

25. If the number of permutations of 6 things is 360, how many are taken at a time?

26. If ${}_nP_2 = 30$, what is n?

27. If ${}_nP_n = 40{,}320$, find n.

28. If ${}_nP_6 = 10 \times {}_nP_5$, find n.

29. If ${}_nP_5 = 12 \times {}_nP_3$, find n.

30. If ${}_{11}P_r = 990$, what is r?

31. If ${}_{2n}P_3 = 100 \times {}_nP_2$, what is n?

SECTION II — COMBINATIONS

623. Combinations are the different groups into which things, taken the same number at a time, without reference to the order of arrangement, can be formed.

Thus, the combinations of the letters a, b, c, d, taken two at a time, are ab, ac, ad, bc, bd, cd; and taken three at a time, abc, abd, acd, bcd.

While ab and ba are different permutations, they are the same combination.

624. Theorem. *The number of combinations of n things taken r at a time is*

$${}_nC_r = \frac{n(n-1)(n-2)(n-3)\cdots(n-r+1)}{\lfloor r}.$$

Dem. Let ${}_nC_r$ be the number of combinations of the n things taken r at a time.

By Art. 617 each combination of r things can have $\lfloor r$ permutations. Hence the number of combinations is the number of permutations divided by $\lfloor r$; that is (Art. 616),

$$ {}_nC_r = \frac{n(n-1)(n-2)(n-3)\cdots(n-r+1)}{\lfloor r}. $$

625. Theorem. *The number of combinations of n things taken r at a time is the same as the number of combinations of n things taken $n-r$ at a time.*

Dem. For each combination containing r things there is left a combination containing the remaining $n-r$ things; hence,

$$ {}_nC_r = {}_nC_{n-r}. $$

626. Sch. In numerical applications the formula of Art. 625 often involves much less labor than that of Art. 624.

Thus, in determining the number of combinations of 15 things taken 12 at a time, the formula of Art. 624 gives

$$ {}_{15}C_{12} = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}, $$

while that of Art. 625 gives simply

$$ {}_{15}C_3 = \frac{15 \cdot 14 \cdot 13}{2 \cdot 3}. $$

627. Theorem. *If p specified things are to be included in each combination, the number of combinations is the same as of $n-p$ things taken $r-p$ at a time.*

Dem. The only combinations possible are those arising from the combinations of the $n-p$ things, of which $r-p$ are in each combination.

628. Theorem. *The total number of combinations of n things taken any number at a time is $2^n - 1$.*

Dem. Each thing may be treated in two ways, as it may be included or excluded. Either way of treating any one of the

things may be combined with either way of treating each of the other things, giving as the number of such combinations

$$2 \times 2 \times 2 \times \cdots \text{ to } n \text{ factors} = 2^n.$$

But this includes the case in which all the things are excluded. Hence the total number of combinations is $2^n - 1$.

629. Cor. $\quad {}_nC_1 + {}_nC_2 + {}_nC_3 + \cdots {}_nC_r = 2^n - 1.$

630. Theorem. *The total number of combinations of n things taken any number at a time, when p of one kind are alike, q of another kind alike, r of another kind alike, and so on, is*

$$(p+1)(q+1)(r+1)\cdots - 1.$$

Dem. The p like things may be treated in $p+1$ ways, as all may be excluded, or 1, 2, 3, $\cdots$ or p included.

Similarly the q like things may be treated in $q+1$ ways, the r like things in $r+1$ ways, and so on. Hence the total number of ways of treating all the things is $(p+1)(q+1)(r+1)\cdots$.

But this includes the case in which all the things are excluded. Hence the total number of combinations is

$$(p+1)(q+1)(r+1)\cdots - 1.$$

631. Cor. *The total number of combinations of n things taken any number at a time, when p of one kind are alike, q of another kind alike, r of another kind alike, and so on, while the remaining t things are unlike, is*

$$(p+1)(q+1)(r+1)\cdots(2^t) - 1.$$

This follows from Arts. 630 and 628.

632. Theorem. *The number of combinations of $p+q+r+\cdots$ things with p things in each combination of one set, q in each of another, r in each of another, etc., is*

$$\frac{\lfloor p+q+r+\cdots}{\lfloor p \times \lfloor q \times \lfloor r \times \cdots}.$$

Dem. Multiplying numerator and denominator of the formula of Art. 624 by $\lfloor n-r$, it becomes

$$ {}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)\cdots 3\cdot 2\cdot 1}{\lfloor r(n-r)\cdots 3\cdot 2\cdot 1} = \frac{\lfloor n}{\lfloor r \times \lfloor n-r}. $$

Since for each combination of p things there is left a combination of $n-p$ things (Art. 625), the number of combinations of $p+q$ things, with p things in each combination of one set and q in each of another, is, by the above formula,

$$ \frac{\lfloor p+q}{\lfloor p \times \lfloor q}. $$

If there are $p+q+r$ things, the number of combinations, with p things in each combination of one set and $q+r$ in each of another, is

$$ \frac{\lfloor p+q+r}{\lfloor p \times \lfloor q+r}. $$

But the number of combinations of $p+q$ things, with p things in each combination of one set and q in each of another, is

$$ \frac{\lfloor q+r}{\lfloor q \times \lfloor r}; $$

hence the whole number is

$$ \frac{\lfloor p+q+r}{\lfloor p \times \lfloor q+r} \times \frac{\lfloor q+r}{\lfloor q \times \lfloor r} = \frac{\lfloor p+q+r}{\lfloor p \times \lfloor q \times \lfloor r}. $$

As the same reasoning applies to any number of sets of combinations, the formula is as stated in the theorem.

633. Cor. *If there are m sets of combinations, and $p=q=r=\cdots$, the above formula becomes*

$$ \textit{either } \frac{\lfloor mp}{(\lfloor p)^m} \quad \textit{or} \quad \frac{\lfloor mp}{\lfloor m \times (\lfloor p)^m}, $$

according as the combinations of the different sets are treated as distinct or identical.

EXAMPLES CL

1. How many different couples can be selected from 6 people?

2. How many different sums of 3 figures each can be formed from the digits 1, 2, 3, 4, 5?

3. How many different products of 2 factors each can be formed from the nine digits? How many of 3 factors each? Of 4 factors each? Of 5 factors each? Of 6 factors each? Of 7 factors each? Of 8 factors each? Of 9 factors each?

4. How many sets of fours can be formed, at different times, from 6 soldiers?

5. How many different committees of 5 each can be formed out of a corporation of 12 members?

6. From a company of 50 soldiers, how many pickets of 6 men each can be formed?

7. In how many ways can 4 vacancies be filled from 10 applicants?

8. How many span can be formed from 10 horses?

9. A druggist accepts an offer of $ 5 for as many glasses of soda water as can be flavored with any two of his 20 sirups. Does he gain or lose by the transaction, and how much, the price being 5 cents per glass?

10. In how many ways can a base-ball nine be selected from 15 players, the pitcher and the catcher being always the same men?

11. How many different amounts can be weighed with 5 weights of 1 pound, 2 pounds, 4 pounds, 8 pounds, 16 pounds, respectively?

12. How many different products can be formed from 5 different factors, taken any number at a time?

QUERY. Why must 5 of the combinations be rejected?

13. In how many different ways can 12 stops of an organ, any number at a time, be opened?

14. How many different sums can be formed with a three-cent piece, a five-cent piece, a dime, a quarter dollar, a half dollar, and a dollar? Would the answer be the same with any six pieces of money of different values?

15. In how many ways can a selection from 5 rubies, 4 diamonds, and 3 emeralds, taking at least one of each kind, be made?

16. In how many ways can a selection from 5 pears and 6 apples, taking at least one of each kind, be made?

17. With 5 dimes and 5 half dimes, how many different amounts could be put into a contribution box?

18. With 3 weights of one denomination, 5 of another, and 4 of another, how many different amounts can be weighed?

19. From 5 apples, 4 apricots, an orange, a pear, a peach, and a banana, how many choices of fruit may be made?

20. From a committee of 10, how many subcommittees of 2, 3, and 5 members respectively can be appointed?

21. From 12 soldiers, how many different scouting parties of 2, 4, and 6, respectively, could be formed?

22. In how many different ways can the 52 cards of a pack be divided equally (a) among 4 players, (b) into 4 piles on the table?

Sug. When divided among the players, the mere exchange of what they hold would give different hands to the respective players; but when divided into piles on the table, the exchange of positions would not give different piles. See Art. 633.

23. In how many different pairs can a railroad section boss distribute his 10 men for work along the track?

Query. In what way does this differ from example 1?

ANSWERS

Examples I

2. $2m^2 + 2mn + 2n^2$. **3.** $6ab - 9a^2x + 7ax^2 + ax^3$. **4.** x^4.

5. $15a^2cx^2 + 2a^2bx^2 + 7mx^2y^2$. **6.** $ab^2 - 2a^2b + a^3 - 6ac - 4ac^2 - c^3$.

7. $a^2 - xy$. **8.** $a^2b^3 + x^2y$. **9.** $4a^2 - b^2 + a^2b + ab^2 + 2b^3 - 3a^3$.

10. $6m^2 + 2n^2 + 2\frac{2}{3}m^2n + 1\frac{1}{2}mn^2$. **11.** $4x^{\frac{1}{2}} - x^{\frac{1}{3}} + 5x^2 + 5x^2y - ab - x^3 - 3$.

12. $5cz - \sqrt{x} + 2\sqrt{z}$.

Examples II

1. $(a + bc - 3b - 2c + 4)x$. **2.** $2bx + (c - 5b)y$.

3. $(a+4)x + (2b-3)y + (c-4)z$. **4.** $(a+m)(x+y) + (b-n)(x-y)$.

5. $2a$. **6.** $(2+4a)\sqrt{x-y}$. **7.** $\frac{5}{6}\sqrt[3]{a^2-x}$. **8.** $\dfrac{a}{\sqrt{x}} + \dfrac{3-m}{y} + 3b$.

9. $5a\sqrt{m^2-x^2}$. **10.** $(a+b+c)\sqrt[5]{x^2-y^2}$.

Examples III

2. $x^3 + x^2 - 2x - 8$. **3.** $4xy$. **4.** $6x + 2x^3$. **5.** $3x^3 - 13x^2 + 9x - 3$.

6. $2a^3 - 2a^2 + 4a$. **7.** $4a^2$. **8.** $x^4 + 5x^3y - 2x^2y^2 - 9xy^3$.

9. $3a + 3b + 3d + 3c$. **10.** $6\sqrt{x} + 2\sqrt{x^3}$. **11.** $4x^{\frac{2}{3}}y^{\frac{2}{3}}$.

12. $3(x+y) + 6(x-y)$. **13.** $2b\sqrt{1-c^2}$. **14.** $\frac{5}{8}\sqrt[3]{a+x^2}$.

15. $2(a-b+c)\sqrt{x^2+y^2}$. **16.** $(a-b)\sqrt{x+y} + (a+b)\sqrt{x-y}$.

17. $3x^2 + 7x - 8$. **18.** $2x^3 - 4x^2y - 5xy^2 + 3y^3$. **19.** $3a - 6b + 4c$.

20. $6\sqrt[3]{x+y} + (a-b)\sqrt{x+y} + (c-d)(x-y)$.

Examples IV

1. $3x^2 - 3x - 5$. **2.** $6a - 4b - 2$. **3.** $3a + 2$. **4.** $4x^2 - 4x + 5$.

5. $6m + 2$. **8.** $-(3-a)x^3 - (c-2b-4)x^2 - (2d-4)x$.

6. $2z$. **9.** $-(c-a^2)x^3 - (b-a+5)x^2$.

7. $x + y + z$. **10.** $-(b-a)x^3 - (b+2c)x^2 - (b+c+d)x$.

Examples V

5. $20\,a^5b^5$. **6.** $72\,ab^{-1}c^3$. **7.** $-42\,x^3y^{\frac{5}{4}}$. **8.** $160\,x^{\frac{7}{6}}y^{-\frac{1}{6}}$. **9.** $30\,a^6b^7c^3$.

10. $72\,a^3bx^5y^6$. **11.** $-42\,x^{\frac{5}{3}}y^2$. **12.** $-216\,a^3bc^{-\frac{1}{6}}$. **13.** $-180\,x^{3m+n+1}y^{m+4n+4}$.

Examples VI

2. $6\,a^4-23\,a^3b+41\,a^2b^2-42\,ab^3+18\,b^4$. **3.** $3\,x^4-7\,x^3-4x^2+16\,x-8$.

4. $8\,x^5-6\,x^4-25\,x^3+13\,x^2+18\,x-8$. **5.** x^5+y^5. **6.** $x^4+x^2y^2+y^4$.

7. $m^6-3\,m^2n^2p^2+n^6+p^6$. **8.** $a^{2m}-a^{m+n}+a^{m+2}-a^{m+1}+a^{n+1}-a^3$.

9. $8\,a^3+27\,b^3-c^3+18\,abc$. **10.** $a^3+b^3+c^3-3\,abc$.

12. $2\,x^7+10\,x^6-9\,x^5+21\,x^4-22\,x^3-11\,x^2+24\,x-12$.

13. $6\,x^6-3\,x^5y+x^4y^2-11\,xy^5+15\,y^6$.

14. $m^8-10\,m^6+5\,m^5+25\,m^4-43\,m^3-12\,m^2+44\,m-40$.

15. $x^7+x^6+x^5+16\,x^4-24\,x^2+41\,x-15$.

16. $3\,x^9+10\,x^7-4\,x^5-9\,x^4+16\,x^3+6\,x^2-12$.

17. $8\,x^{4p}+6\,x^{3p}y^q+3\,x^{2p}y^{2q}+22\,x^py^{3q}+6\,y^{4q}$.

Examples VII

3. $9\,x^4+3\,x^3-2\,x^2+6\,x-4$. **4.** $x^5-5\,x^4+10\,x^3-10\,x^2+5\,x-1$.

5. $6\,a^4-10\,a^3x-22\,a^2x^2+46\,ax^3-20\,x^4$. **6.** $81\,x^4-y^4$.

7. $6\,x^6+10\,x^5-38\,x^4+11\,x^3+46\,x^2-17\,x-15$. **8.** $m^5+4\,mn^4$.

9. $x^6-50\,x^4+769\,x^2-3600$. **10.** x^8+x^4+1.

11. $x^5-5\,ax^4+10\,a^2x^3-13\,a^3x^2+13\,a^4x-6\,a^5$.

12. $x^6+3\,x^5y-3\,x^4y^2-11\,x^3y^3+6\,x^2y^4+12\,xy^5-8\,y^6$.

13. $2\,x^7-11\,x^6+14\,x^5+17\,x^4-50\,x^3+23\,x^2+38\,x-30$.

14. $x^7-7\,x^6+21\,x^5-17\,x^4-25\,x^3+6\,x^2-2\,x-4$.

Examples VIII

2. $x^4-x^3y-13\,x^2y^2-40\,xy^3+25\,y^4$.

3. $2\,x^5+7\,x^4y+3\,x^3y^2+8\,x^2y^3-5\,xy^4+300\,y^5$.

4. $4\,m^6+12\,m^5n-24\,m^4n^2+6\,m^3n^3-72\,m^2n^4+36\,mn^5-20\,n^6$.

6. $5\,x^7-15\,x^6y-6\,x^5y^2+18\,x^4y^3-4\,x^3y^4+12\,x^2y^5+7\,xy^6-21\,y^7$.

7. $x^5+3\,x^4y+28\,x^3y^2+3\,x^2y^3+12\,xy^4-63\,y^5$.

8. $a^5+8\,a^4b-43\,a^3b^2+67\,a^2b^3+72\,ab^4-144\,b^5$.

10. $2\,x^5+13\,x^4+19\,x^3+25\,x^2+31\,x+30$.

11. $4x^5 - 19x^4 + 10x^3 + 7x^2 + 3x + 4$.

12. $x^6 + 4x^5 - 16x^4 + 37x^3 + 8x^2 - 50x - 56$.

13. $x^6 + 6x^5 - 8x^3 - 48x^2 + 5x + 30$. **14.** $3x^5 - 27x^4 + 7x - 63$.

15. $5x^6 + 60x^5 - 5x - 60$. **16.** $x^7 - 10x^6 + 3x^4 + 30x^3 + 6x - 30$.

Examples IX

2. $x^5 - 3x^4 - 6x^3 + 12x^2 - 40x + 96$. **3.** $x^4 - 9x^3 + 16x^2 - 9x - 35$.

4. $x^4 + 4x^3 - 19x^2 - 46x + 120$.

5. $2x^6 + 3x^5 - 25x^4 - 13x^3 + 11x^2 - 26x + 168$.

6. $x^5 - 9x^4 + x^3 + 105x^2 - 74x - 168$.

7. $3x^6 - 7x^5 - 51x^4 + 41x^3 + 96x^2 - 142x + 60$.

8. $9x^6 - 18x^5 - 193x^4 + 206x^3 + 444x^2 - 88x - 160$.

9. $x^7 - 2x^6 - 10x^5 + 28x^4 + 5x^3 - 74x^2 + 76x - 24$.

Examples X

5. $x^2 + 4x - 60$. **6.** $x^2 - 4x - 96$. **7.** $m^2 + 16m - 80$.

8. $m^2 + 19m + 48$. **9.** $x^2 - 18x + 77$. **10.** $y^2 - 18y - 88$.

11. $y^2 + 80y - 900$. **12.** $x^2 + 25x - 1250$. **14.** $x^4 - 14x^2 + 48$.

15. $x^4 - 5x^2 - 84$. **16.** $x^6 + 9x^3 - 70$. **17.** $z^8 - 25z^4 - 150$.

20. $m^2x^6 - 10mx^3 - 39$. **21.** $m^4z^8 + 9m^2z^4 - 90$. **22.** $a^6x^4 - 16a^3x^2 - 80$.

23. $a^6x^6 + 20a^3x^3 - 69$. **25.** $4x^2 - 4x - 63$. **26.** $16x^4 - 12x^2 - 88$.

27. $9z^6 - 18z^3 - 72$. **28.** $25z^4 - 100z^2 + 75$. **30.** $x^2 - 16xy + 60y^2$.

31. $x^2 + 16xy + 48y^2$. **32.** $x^2 - 16xy + 63y^2$. **33.** $x^4 + 4x^2y - 32y^2$.

34. $4z^4 + 8z^2y - 45y^2$. **35.** $z^4 - 4z^2y^3 - 32y^6$. **36.** $z^6 - 4z^3y^2 - 77y^4$.

37. $9x^6 + 15x^3y^2 - 14y^4$. **39.** $x^2 - (8y - 3z)x - 24yz$.

40. $x^4 + (11y - 4z)x^2 - 44yz$. **41.** $9x^6 + 3(7y - 5z)x^3 - 35yz$.

42. $25a^4 + 5(6m^3 - 3n^2)a^2 - 18m^3n^2$.

Examples XI

1. $1 + 2x + x^2$. **2.** $x^2 + 4x + 4$. **3.** $x^4 + 2x^2y + y^2$. **6.** $x^{2n} + 4x^n + 4$.

7. $\dfrac{x^2}{y^2} \pm 2 + \dfrac{y^2}{x^2}$. **8.** $\frac{4}{9}x^{\frac{1}{2}} - 1 + \dfrac{9}{16x^{\frac{1}{2}}}$. **9.** $x^{\frac{3}{2}} + \dfrac{x^{\frac{3}{4}}}{3y^{\frac{1}{2}}} + \dfrac{1}{36y}$.

12. $a^{\frac{1}{2}} - ab^3 + \frac{1}{4}a^{\frac{3}{2}}b^6$. **13.** $x^{4n} - 2x^{2n}y^n + y^{2n}$. **16.** $(16 + 56x^n + 49x^{2n})x^2$.

17. $49x^3 - 42x^{\frac{3}{2}}y^{-\frac{5}{2}} + 9y^{-5}$. **18.** $\dfrac{4a^{2m}}{9b^{2n}} \pm 2 + \dfrac{9b^{2n}}{4a^{2m}}$. **19.** $x^4 - 4y^2$.

20. $9m^4 - 25n^6$. **21.** $1 - \frac{4}{9}a^2$. **22.** $25x^6 - 9y^4z^2$. **23.** $4x^{\frac{2}{3}} - 9y$.

24. $(81ax - 1)ax$.

Examples XII

1. $a^2 + b^2 + c^2 - 2\,ab - 2\,ac + 2\,bc$. 2. $a^4 + b^4 + c^2 - 2\,a^2b^2 + 2\,a^2c - 2\,b^2c$.

3. $m^2 + n^2 + p^2 + q^2 - 2\,mn - 2\,mp + 2\,mq + 2\,np - 2\,nq - 2\,pq$.

4. $x^4 + 4\,y^2 + u^2 + 9\,v^2 + 4\,x^2y - 2\,x^2u - 6\,x^2v - 4\,yu - 12\,yv + 6\,uv$.

5. $x^6 + 2\,x^5 + 3\,x^4 + 4\,x^3 + 3\,x^2 + 2\,x + 1$.

7. $x^4 - 2\,x^3y + 3\,x^2y^2 - 2\,xy^3 + y^4$.

8. $x^6 + 2\,x^5y + 3\,x^4y^2 + 4\,x^3y^3 + 3\,x^2y^4 + 2\,xy^5 + y^6$.

9. $9\,x^4 + 25\,y^2 + 16\,z^6 + 4\,u^2 - 30\,x^2y + 24\,x^2z^3 - 12\,x^2u - 40\,yz^3 + 20\,yu - 16\,z^3u$.

10. $a^8 + 4\,b^6 + 9\,c^4 + 16\,d^2 - 4\,a^4b^3 - 6\,a^4c^2 + 8\,a^4d + 12\,b^3c^2 - 16\,b^3d - 24\,c^2d$.

Examples XIII

2. $x^2 - 6\,xy + 9\,y^2 - 25\,z^2$. 3. $x^4 + x^2y^2 + y^4$. 4. $4\,x^2 - 4\,xz + z^2 - 16\,y^2$.

5. $16\,x^8 + 23\,x^4y^4 + 9\,y^8$. 7. $x^4 - 2\,x^3 + x^2 - 81$. 8. $x^4 - x^2 + 14\,x - 49$.

9. $9\,x^4 + 14\,x^2 + 25$. 10. $a^2 + 4\,ab + 4\,b^2 - 9\,c^2 - 6\,cd - d^2$.

Examples XIV

1. $2\,a^2$. 2. $5\,x^2$. 3. $4\,x^2z^{-2}$. 4. $3\,x^{m-n}$. 5. $6\,x^{m+n}$. 6. $a^2x^{\frac{1}{6}}$.

7. $a^{-1}x^{\overline{\frac{m-n}{n}}}$. 8. $(ab)^{3m}$. 9. $6\,x^{-5}$. 10. $m^2x^{\frac{4}{3}}$. 11. $a^2(a - x)^2$.

12. $4\,m(a^2 - 3\,x)^5$. 13. $3^3(x^2 - y^2)$. 14. $a^2(2\,x + 4\,x^3)^n$.

15. $\dfrac{a^2x^2}{b^3y^3}$. 16. $\dfrac{3\,a^3d^3x^5}{5\,b^2c}$. 17. $\dfrac{4\,x^{m+n}}{7\,y^{m+n}z^p}$.

Examples XV

1. $2\,ab^2 + 5\,a^3b - 4\,ab$. 2. $4\,x^2y - 12\,y^2 + 9$. 3. $-3\,x^2 + 4\,ax - a^2$.

4. $5\,a^2 - 3\,b^2 - 4\,c^2$. 5. $3\,x^m + 2\,x^{2m} - 4\,x^{3m}$. 6. $2\,a^m\,b^{s-2n} - 3\,a^{m+2n}\,b^s$.

7. $7\,a^{20}b^4 - b^8 + 13\,a^{12}b^6$. 8. $x^{\frac{4}{3}} + 3\,x^{\frac{4}{3}}y^2 - 2$. 9. $a^{1-2n} - a^{1-n} - a + a^{1+n}$.

10. $2(x - y) - 3(x - y)^2 + 4(x - y)^3$.

Examples XVI

2. $5\,x^2 - 4\,x + 3$. 3. $2\,x^2 - 7\,x - 8$. 4. $2\,a^3b + 3\,a^2b^2 - ab^3 + 4\,b^4$.

Examples XVII

1. $5\,a^2 - 6\,ax - 2\,x^2$. 2. $x^2 - 5\,ax + 4\,a^2$. 3. $2\,y^4 - 8\,y^3 + 12\,y^2 - 8\,y + 2$.

4. $x^2 - 3\,x + 5$. 5. $2\,a^3b + 3\,a^2b^2 - ab^3$. 6. $a^3 + 3\,a^2x + 3\,ax^2 + x^3$.

7. $9\,x + 6\,y + 3\,z$. 8. $2\,y^3 - 4\,xy^2 + x^2y + 3\,x^3$. 9. $3\,x^3 + x^2y - 4\,xy^2 + 2\,y^3$.

10. $1 - x + x^3 - x^4 + 2x^5$. **11.** $4x^2 + 6x + 9$. **12.** $-x^2 - 2x - 4$.
13. $2a^2 - ab + 2b^2$. **14.** $7x^2 + 5xy + 2y^2 + \dfrac{7}{2x^2 + 5xy + 7y^2}$.
16. $x + y$.
17. $a^4 - a^3b + a^2b^2 - ab^3 + b^4$. **18.** $x^4 + 3x^3y + 8x^2y^2 - 8y^4$.
19. $\frac{3}{8}a^2 - \frac{1}{4}a - \frac{2}{3}$. **20.** $6x - \frac{1}{3}y - \frac{1}{2}$. **21.** $ax^2 - 2x + b$. **23.** $3x^2 + 2x + 1$.
24. $x^2 - 3x + 7$. **25.** $2a^3 - 6a^2b + 18ab^2 - 27b^3$. **26.** $2x^3 - x + 1$.
27. $2a^3 + 4a^2 + 8a + 16$. **28.** $a + b$. **29.** $m^m + an^{an}$.
30. $2a^3 - 3a^2b + ab^2 + 4b^3$.
31. $2x^4 - 3x^3y + 4x^2y^2 - 5xy^3 + 6y^4$, with remainder $4y^5$.

Examples XVIII

2. $7x^2 + 11xy + 20y^2$. **3.** $x^3 + 6x^2y - xy^2 - 30y^3$.
4. $2x^3 + 3x^2y - 3xy^2 + 4y^3$. **5.** $12x^3 + 11x^2y^2 - y^3$, with remainder $3y^4$.
6. $x^4 + x^3y - 5x^2y^2 - 3xy^3 + 3y^4$. **7.** $3x^4 + 8x^3y - 5x^2y^2 - 24xy^3 - 24y^4$.
8. $3x^4 + 2x^3y + 3x^2y^2 - 4xy^3 + 2y^4$.
9. $4x^5 - 4x^4y + 3x^3y^2 + x^2y^3 + 6xy^4 + 5y^5$.
10. $x^5 - 8x^4y + 3x^3y^2 + 4x^2y^3 + 2y^5$. **12.** $x^3 + 8x^2 + 6x + 2$.
13. $3x^4 - 9x^3 + 27x^2 + 4x - 15$. **14.** $8x^2 - 40x + 100$.
15. $5x^3 - 6x^2 + 7$. **16.** $x^4 + 5x^3 + 5x^2 + 5x + 5$.
17. $x^4 - x^3 + 3x^2 + 6x - 3$. **18.** $2x^3 + 5x^2 + 5x$.
19. $x^5 + 5x^4 + 3x^3 - 4x^2 - 2x + 5$. **20.** $x^4 - 2x^3 + x^2 + 2x + 3$.
22. $x^3 + 3x^2 + 9x + 27$. **23.** $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.
24. $x^5 + 3x^4y + 9x^3y^2 + 27x^2y^3 + 81xy^4 + 243y^5$.

Examples XIX

2. $x^2 - 3x + 2$. **3.** $x + 7$. **4.** $x^3 - 3x^2 + 2x - 3$. **5.** 1. **7.** $x^2 - 6x - 2$.

Examples XXI

2. $x^2 + 3x + 2$. **3.** $x^2 + 3x + 2$. **4.** $x^4 + x^2 - 4x - 4$.
5. $x^4 + 4x^3 + 6x^2 + 4x + 1$. **6.** $x^3 - 2x^2 + 3x - 5$. **7.** $3x^3 + 4x^2 + 5x + 7$.

Examples XXII to XXVIII

The student would derive little benefit from these examples if the answers were given.

Examples XXIX

2. $12a^2b^2$. **3.** $3x^2y^2$. **4.** $3m^2(x-y)^3$. **5.** $x^2y^2(2x+y)$. **6.** $x + 2$.
7. $x - 2$. **8.** $x - 1$. **9.** $x + 1$. **10.** $2x + 1$. **11.** $a + b$. **12.** $x^2 - x$.
13. $x^2 + 2xy$. **14.** $x + 6$. **15.** $x - 3$. **16.** $x^2 + 4x + 3$. **17.** $x + 2$.

18. x^2-2x+4. **19.** $x-1$. **20.** x^2-3x-4. **21.** x^2+3x-4.
22. x^2-6x+8. **23.** x^2-x+1. **24.** $4(x^2-2xy+y^2)$. **25.** $a-2b$.
26. $a^2-3ab-4b^2$. **27.** $3x-4$. **29.** $x^3-6x^2+11x-6$.
30. $x^3+5x^2-2x-24$. **31.** $x^4+4x^3-2x^2-12x+9$.
32. $3x^4+4x^3-7x^2-4x+4$. **33.** $2x^3-3x^2+4x-5$.

Examples XXX

2. x^2+x+1. **3.** $3x-4$. **4.** $x^2+xy+3y^2$.
5. $x^4-x^3y+x^2y^2-xy^3+y^4$.

Examples XXXI

2. $3x-3$. **3.** $7x+1$. **4.** $2x^2-3x+3$. **5.** $2x^2-3x+1$.

Examples XXXII

1. $x+6$. **2.** $x^2-8x+15$. **3.** $2(x+y)$.

Examples XXXIII

2. x^2-4. **3.** $(x+y)^3(x-y)^2$. **4.** $x^3+7x^2-9x-63$.
5. $x^3-9x^2-x+105$. **6.** $x^4-10x^3-39x^2+460x-700$.
7. $x^4+9x^3+11x^2-81x-180$. **8.** x^3-2x^2-5x+6.
9. $(6x^2-7x-5)(x-4)(2x+4)$. **10.** $a^5+2a^4-10a^3-20a^2+9a+18$.
11. x^4-16y^4. **12.** $x^4+5x^3+5x^2-5x-6$. **13.** a^4-1.
14. $x^5+7x^4-10x^3-70x^2+9x+63$.
15. $(2x-3y+4)(3x+2y-5)(4x-y+3)$.

Examples XXXIV

1. $\frac{5b}{7c}$. **2.** $\frac{2ac^n}{3b}$. **3.** $\frac{7a^{\frac{1}{6}}b}{11c^{2n}}$. **4.** $\frac{x}{a}$. **5.** $\frac{1}{x^2+y^2}$. **6.** $\frac{3a}{4b}$.
7. $\frac{x+2}{x-1}$. **8.** $\frac{x+3}{x+5}$. **9.** $\frac{1}{x+1}$. **10.** $\frac{x-2}{x+3}$. **11.** $\frac{1+x}{(1-x)^2}$.
12. $\frac{a^2-ax+x^2}{a+x}$. **13.** $\frac{x^2+3x+9}{x+1}$. **14.** $\frac{x-5}{x-3}$. **15.** $\frac{1-y}{1+y}$.
16. $\frac{1-2a}{1-4a}$. **17.** $\frac{x-y}{y}$. **18.** $\frac{a+b-c}{a+b+c}$. **19.** $\frac{a+b-c-d}{a-b+c-d}$.
20. $\frac{2x+1}{2x+3}$. **21.** $\frac{a+b+c+d}{b-a-c-d}$. **22.** $\frac{x+3}{x-3}$. **23.** $\frac{x-3}{x-5}$.
24. $\frac{2x+5}{7x-5}$. **25.** $\frac{x-3}{x(3x-7)}$. **26.** $\frac{x+4}{x-5}$. **27.** $\frac{2x+3}{3x-1}$. **28.** $\frac{3x-5}{4x-3}$.

Examples XXXV

2. $2x - 4 - \frac{2}{3x}$.
3. $3x^2 - 2x + 2 + \frac{7}{x-2}$.
4. $2x^2 + 2xy + 10y^2 + \frac{5}{x - 3y}$.
5. $m + n - \frac{x+y}{m+n}$.
6. $3x - 2 - \frac{5x - 1}{4x^2 + 3}$.
7. $16x^4 - 24x^3 + 36x^2 - 54x + 81$.
8. $2x - 1 + \frac{x - 6}{x^2 - x - 1}$.
9. $4x^2 + 6x - 2 - \frac{8x - 5}{2x^2 + x - 3}$.
10. $5x - 14 + \frac{31}{x+2}$.
11. $x + y$.

Examples XXXVI

1. $\frac{b^3}{a}$.
2. $\frac{18x^2}{3x-4}$.
3. $-\frac{4ax}{a-x}$.
4. $\frac{x^3 - y^3}{x+y}$.
5. $\frac{a^4 + b^4}{a-b}$.
6. $\frac{(a+b+c)(b+c-a)}{2bc}$.
7. $\frac{x^2 + 3x + 1}{x+5}$.
8. $\frac{a(a+b)}{a-b}$.

Examples XXXVII

1. $\frac{x(a-b)}{a^2-b^2}, \frac{y(a+b)}{a^2-b^2}, \frac{z}{a^2-b^2}$.
2. $\frac{15}{6(1-x^2)}, \frac{6}{6(1-x^2)}, -\frac{4+4x}{6(1-x^2)}$.
3. $\frac{x^4 - x^3y + x^2y^2 - xy^3}{x^4 - y^4}, \frac{x^4 + x^2y^2}{x^4 - y^4}, \frac{x^3}{x^4 - y^4}$.
4. $\frac{x(a^2-b^2)}{(a^2-b^2)^2}, \frac{y(a-b)^2}{(a^2-b^2)^2}, \frac{z(a+b)^2}{(a^2-b^2)^2}$.
5. $\frac{2x(x+y)}{x^3+y^3}, \frac{3y}{x^3+y^3}, \frac{4z(x^2-xy+y^2)}{x^3+y^3}$.
6. $\frac{(x-y)^2}{(x-y)^3}, \frac{x(x-y)}{(x-y)^3}, \frac{x^2}{(x-y)^3}$.
7. $\frac{x+2}{(x+3)(x^2-4)}, \frac{2(x-2)}{(x+3)(x^2-4)}, \frac{3(x+3)}{(x+3)(x^2-4)}$.
8. $\frac{(x+2)^2}{x^3+6x^2+12x+4}, \frac{x+2}{x^3+6x^2+12x+4}, \frac{1}{x^3+6x^2+12x+4}$.
9. $\frac{(x-2)^2(x+1)^2}{x^4-5x^2+4}, \frac{(x+2)^2(x-1)^2}{x^4-5x^2+4}, \frac{(x+2)^2(x+1)^2}{x^4-5x^2+4}, \frac{(x-2)^2(x-1)^2}{x^4-5x^2+4}$.

Examples XXXVIII

2. $\frac{2x}{x^2-y^2}$.
3. $\frac{2}{x-x^3}$.
4. $\frac{2b^3}{b^2-a^2}$.
5. $\frac{1}{2x-x^2}$.
6. $\frac{5x+1}{x^4-1}$.
7. $\frac{2x-a}{x+a}$.
8. $\frac{x-5}{(x-3)(x-4)}$.
9. $\frac{x+4}{x-4}$.
10. $\frac{1}{x^3-1}$.

11. $\dfrac{x+2}{(1-x)(x-3)}$. 12. $\dfrac{1}{1+2x}$. 13. 0. 14. $-\dfrac{3(x^2+1)}{x^4+x^2+1}$.

15. 0. 16. $\dfrac{3x^2+2x+13}{x^3+27}$. 17. $\dfrac{1}{x-3}$.

Examples XXXIX

2. $\dfrac{9a^2b^5y}{4x^3}$. 3. $\dfrac{a}{x+y}$. 4. $\dfrac{x+4}{x-3}$. 5. $\dfrac{3}{x-2}$. 6. $\dfrac{b^2c}{a^3}$.

7. $\dfrac{a}{(a^2-b^3)^2}$. 8. $\dfrac{3x+1}{x-5}$. 9. $a^2+1+\dfrac{1}{a^2}$. 10. $\dfrac{2x^4+6x^3-36x^2}{(x-6)(x^2-6)}$.

11. $\frac{1}{2}$. 12. $\dfrac{1-y}{x}$. 13. 1. 14. $\dfrac{y}{x(x^2-y^2)}$. 15. $\dfrac{1}{x+y}$.

16. $\dfrac{x-3y}{x+3y}$. 17. $\dfrac{x^2-y^2}{2(x^2+y^2)}$. 18. $\dfrac{x+3}{x-1}$. 19. $\dfrac{3x+1}{x+3}$.

20. $\dfrac{x^2+6x+8}{x^2+8x+15}$. 21. 1.

Examples XL

1. $\dfrac{5ab^2}{12xy^2}$. 2. $\dfrac{x^2-3y}{x+4y}$. 3. $\dfrac{x-y}{3ax}$. 4. $\dfrac{a^n-b^m}{ab}$. 5. $3(a+b)$.

6. $\dfrac{x(x+7)}{(x-3)^2}$. 7. $\dfrac{x^2+xy+y^2}{x^2-xy+y^2}$. 8. $\dfrac{x-a}{x+a}$. 9. $\dfrac{x(x+2y)}{y}$.

10. $\dfrac{2(a-b)^2}{3b^2(a+b)}$. 11. $a^2-2ac+c^2-b^2$. 12. $\dfrac{2x-1}{2x-3}$. 13. $\dfrac{x+1}{x+5}$.

14. $\dfrac{2a+x}{a+2x}$. 15. 1. 16. 1. 17. 1. 18. $\dfrac{x+3}{x-3}$.

Examples XLI

1. $\dfrac{adfh-bcfh}{bdeh+bdfg}$. 2. $a-b$. 3. $\dfrac{b}{a}$. 4. $\dfrac{a^2+1}{2a}$. 5. $\dfrac{x-y}{x+y}$.

6. $\dfrac{a+3}{a-2}$. 7. $\dfrac{a^2+b^2+c^2}{a^2b^2+a^2c^2+b^2c^2}$. 8. $\dfrac{2(m-n)}{(m+n)^2}$.

Examples XLII

1. $\dfrac{xyz+x+z}{yz+1}$. 2. $\dfrac{adf+ae}{bdf+be+cf}$. 3. $\dfrac{x^2+x-1}{x^2-3}$.

4. 1. 5. $(a^3-b^3)^2$. 6. $\dfrac{4}{3(x+1)}$.

Examples XLIII

1. $\dfrac{xy^nc^4}{a^2b^3}$. 2. $\dfrac{ab^3-a^3}{b^3+a^3b}$. 3. $\dfrac{x^3y^3}{(x-y)^3(x^3+y^3)}$.

4. $\dfrac{abxz(a-b)}{y^3(a^2-ab+b^2)}$. 5. $\dfrac{x^4(y^2-x)^2}{y(y+x^2)^3}$. 6. $\dfrac{(1-x^2)^p}{x^{m+2p}(x^2-y)^n}$.

Examples XLIV

1. $\sqrt{a^3}\sqrt[3]{b^2}$. 2. $\sqrt{a}\sqrt[3]{x^2}$. 3. $\dfrac{5}{\sqrt[5]{(ax)^3}}$. 4. $\dfrac{2\sqrt{x}}{3}$. 5. $\dfrac{5}{7\sqrt[4]{x^3}\sqrt[3]{y^2}}$.

6. $\dfrac{\sqrt{7}}{\sqrt[3]{6}\sqrt[n]{x}\sqrt[p]{y^q}}$. 7. $\dfrac{3\sqrt{a^7}}{\sqrt{5}\sqrt[3]{b^7}}$. 8. $\dfrac{\sqrt[5]{x^6}}{5^2\sqrt[3]{y^4}}$. 9. $\dfrac{\sqrt[3]{4}\sqrt[n]{x^{m+n}}}{\sqrt[n]{y^{np+1}}}$.

10. $x^{\frac{3}{2}}y^{\frac{2}{3}}$. 11. $5x^2y^{\frac{1}{2}}$. 12. $\dfrac{7^{\frac{1}{2}}x^{\frac{1}{6}}}{y^{\frac{2}{3}}}$. 13. $\dfrac{6^{\frac{1}{3}}y^{\frac{4}{5}}}{x^3}$. 14. $\dfrac{a^{\frac{3}{4}}x^3}{5^{\frac{1}{2}}y^{\frac{4}{3}}}$.

15. $x^{\frac{s+t}{m}}y^{\frac{s+t}{n}}$. 16. $\dfrac{(a+b)^{\frac{2}{3}}}{(a-b)^{\frac{3}{2}}}$. 17. $(a^2+2b)^{\frac{4}{3}}(x+y)^{\frac{3}{2}}$. 18. $\dfrac{(x^2+y^2)^{\frac{q}{n}}}{(x^2-y^2)^{\frac{p}{m}}}$.

Examples XLV

1. a^6b^4. 2. $a^4b^{-4}c$. 3. $a^3b^{-\frac{1}{2}}c^{\frac{4}{3}}$. 4. $3^6x^9y^{-2}$. 5. $\dfrac{5^4x^2y^3}{z^4}$.

6. $\dfrac{6^{\frac{5}{2}}x^{10}z^4}{y^2}$. 7. $\dfrac{a^6}{b^2c^6}$. 8. $\dfrac{a^3x}{b^6y^4}$. 9. $\dfrac{3^4a^4x^6}{5^4b^4y^5}$. 10. $a^{\frac{2}{3}}x^2y^4$.

11. $\dfrac{5a^2x^6}{y}$. 12. $\dfrac{2ax^2}{3b^{\frac{1}{3}}y}$. 13. $\dfrac{x^6}{5^3y^9}$. 14. $\dfrac{ax^2}{y^4}$. 15. $\dfrac{8a^3c^{\frac{3}{2}}}{b^6}$.

16. $16x^4y^2z^{\frac{2}{m}}$. 17. $\dfrac{\sqrt[3]{x}}{\sqrt[4]{y}}$. 18. $\dfrac{x^py^m}{z^{\frac{p}{m}}}$.

Examples XLVI

2. $x^4+4x^3y+6x^2y^2+4xy^3+y^4$.

3. $a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$.

4. $8x^6-12x^4y+6xy^2-y^3$. 5. $1-4x^2+6x^4-4x^6+x^8$.

6. $a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$.

7. $x^8-4x^6y^2+6x^4y^4-4x^2y^6+y^8$.

8. $x^2+8x^{\frac{3}{2}}y+24xy^2+32x^{\frac{1}{2}}y^3+16y^4$.

9. $1+10x^2+40x^4+80x^6+80x^8+32x^{10}$.

10. $a^2+\dfrac{8a^{\frac{3}{2}}b^2}{x}+\dfrac{24ab^4}{x^2}+\dfrac{32a^{\frac{1}{2}}b^6}{x^3}+\dfrac{16b^8}{x^4}$.

11. $x^6 - 3x^5 + \frac{15}{4}x^4 - \frac{5}{2}x^3 + \frac{15}{16}x^2 - \frac{3}{16}x + \frac{1}{64}$.

12. $a^2 + 8a^{\frac{7}{4}}b^{\frac{1}{2}} + 28a^{\frac{3}{2}}b + 56a^{\frac{5}{4}}b^{\frac{3}{2}} + 70ab^2 + 56a^{\frac{3}{4}}b^{\frac{5}{2}} + 28a^{\frac{1}{2}}b^3 + 8a^{\frac{1}{4}}b^{\frac{7}{2}} + b^4$.

13. $\dfrac{1}{64x^{12}} - \dfrac{3a^2y^3}{8x^{10}} + \dfrac{15a^4y^6}{4x^8} - \dfrac{20a^6y^9}{x^6} + \dfrac{60a^8y^{12}}{x^4} - \dfrac{96a^{10}y^{15}}{x^2} + 64a^{12}y^{18}$.

14. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 + 4x^3z - 12x^2yz + 12xy^2z - 4y^3z + 6x^2z^2$
$- 12xyz^2 + 6y^2z^2 + 4xz^3 - 4yz^3 + z^4$.

15. $8x^2 - 4\sqrt{(x+1)^3(x-1)} - 4\sqrt{(x+1)(x-1)^3} - 4$.

16. $32x^6 + 6x^5\sqrt{x^2-1} - 48x^4 + 20x^3(x^2-1)^{\frac{3}{2}} + 18x^2 + 6x(x^2-1)^{\frac{5}{2}} - 1$.

Examples XLVII

2. $8x^6 - 36x^5 + 114x^4 - 207x^3 + 285x^2 - 225x + 125$.

3. $216x^6 - 432x^5y + 504x^4y^2 - 64x^3y^3 + 168x^2y^4 - 48xy^5 + 8y^6$.

4. $64x^9 - 96x^8 + 192x^7 + 88x^6 - 96x^5 + 366x^4 + 147x^3 - 15x^2 + 225x + 125$.

5. $x^{12} - 6x^{11} + 21x^{10} - 47x^9 + 81x^8 - 108x^7 + 126x^6 - 117x^5$
$+ 99x^4 - 61x^3 + 42x^2 - 12x + 8$.

Examples XLVIII

5. $5x^2 + 2y - z^2$. **9.** $6a^3 - 2b^2 + 3c - 5d$.

Examples XLIX

3. $6x^3 - 3x^2 - x$. **10.** $4x^4 - 4x^2y^2 - 7y^4$.

16. $4x^3 - 5x^2 + 2x - 3$. **20.** $11x^3 - 3x^2y + 5xy^2 + 9y^3$.

28. $4x^6 - 6x^4 + 2x^3 - 3x^2 + 9$. **32.** $3x^6 - 5x^5 - 4x^2 + 2x - 5$.

33. $6x^6 - 4x^5 - 2x^4 + x^3 + 5x^2 - 3x - 7$.

34. $7x^7 + 5x^6 - 3x^5 - x^4 + 8x^3 + 4x^2 - 2x + 6$.

Examples LII

3. $2x^4 - 3x^2 - 1$. **6.** $4x^3 - 2x^2 + 3x + 5$. **9.** $3x^5 - 2x^4 + x^3 + 4x^2 - 3x - 5$.

Examples LIV

2. $4\sqrt{2}$. **3.** $5\sqrt{3}$. **4.** $6\sqrt{2}$.

5. $14\sqrt{2}$. **6.** $15\sqrt{5}$. **7.** $24\sqrt{5}$.

8. $4ab\sqrt{3b}$. **9.** $3a^2b^2\sqrt{7a}$. **10.** $10bc^3\sqrt{5abc}$.

11. $24xy^2\sqrt{3}$. **12.** $21abx^2\sqrt{5bx}$. **13.** $60a^2mx^2\sqrt{2m}$.

14. $2abc^2\sqrt[3]{7b^2}$. **15.** $9xy\sqrt[3]{3y}$. **16.** $15bc\sqrt[3]{5abc^2}$.

17. $12a^{m+1}b^{2n}\sqrt{3a^m}$. **18.** $20a^mb^{n+3}\sqrt[3]{2a^m}$. **19.** $-21a^2b^2c^3\sqrt[3]{4b}$.

20. $22\,xy^2\sqrt{3\,x}$. **21.** $36\,ab^{-2}c^2\sqrt{5\,c}$. **22.** $20\,ab^{-2}c^2\sqrt[3]{7\,c}$.
23. $a\sqrt{a+b}$. **24.** $3\,x\sqrt{2\,x-3\,y}$. **25.** $2\,a^2b\sqrt{5\,a-8\,a^2b}$.
26. $x(x+y)\sqrt{2}$. **27.** $(x^2-xy^2)\sqrt{3\,x}$. **28.** $2\,a\sqrt[3]{3\,a-4\,b}$.

Examples LV

2. $\sqrt{7}$. **3.** $\sqrt{13}$. **4.** $a\sqrt{15\,b}$. **5.** $\sqrt[3]{7}$. **6.** $\sqrt{6}$. **7.** $b\sqrt{10\,a}$. **8.** $\sqrt{3}$.
9. $a\sqrt{5\,b}$. **10.** $\sqrt[3]{4}$. **11.** $-a\sqrt[3]{9\,b^2c}$. **12.** $\sqrt[3]{12\,a^{m+2}}$. **13.** $\frac{2}{b}\sqrt{2\,a}$.
14. $\sqrt[5]{14\,a^3b^2}$. **15.** $3\sqrt[3]{3\,ab^2}$. **16.** $\sqrt{3\,ab}$. **17.** $4\sqrt{x+2\,y}$. **18.** $2\sqrt{x-3\,y}$.

Examples LVI

1. $\sqrt{9\,a^2b^4}$, $\sqrt{49\,x^6y^4}$, $\sqrt{121\,x^2y^8}$, $\sqrt{4\,x^2-12\,xy^2+9\,y^4}$.
2. $\sqrt[3]{64\,x^6y^3}$, $\sqrt[3]{125\,a^3x^9}$, $\sqrt[3]{-27\,a^{-6}b^3c^6}$, $\sqrt[3]{216\,x^3y^6z^{-9}}$. **3.** $\sqrt{18}$.
4. $\sqrt{48\,a^3}$. **5.** $\sqrt{2}$. **6.** $\sqrt{\frac{1}{3}}$. **7.** $\sqrt[3]{81}$. **8.** $\sqrt[3]{\frac{a^3}{5}}$. **9.** $\sqrt[3]{2\,x}$.
10. $\sqrt{x(a+b)}$. **11.** $\sqrt[3]{a-b}$. **12.** $\sqrt{\frac{x-y}{x+y}}$. **13.** $(a^2-b^2)^{\frac{3}{2}}$.
14. $\sqrt{\frac{m+n}{m-n}}$. **15.** $(4\,x^2-4\,y^2)^{\frac{3}{2}}$. **16.** $(x^8-x^2)^{\frac{3}{4}}$. **17.** $\sqrt{x^2-1}$.

Examples LVII

2. $\sqrt[6]{8}$, $\sqrt[6]{9}$. **3.** $\sqrt[15]{32}$, $\sqrt[15]{8}$. **4.** $\sqrt[10]{32}$, $\sqrt[10]{9}$. **5.** $\sqrt[12]{256\,a^8}$, $\sqrt[12]{27\,a^9}$.
6. $\sqrt[mn]{a^{3n}}$, $\sqrt[mn]{b^{4m}}$. **7.** $\sqrt[12]{a^{4m}}$, $\sqrt[12]{b^{3n}}$. **8.** $\sqrt[12]{a^6}$, $\sqrt[12]{b^4}$, $\sqrt[12]{c^3}$.
9. $\sqrt[12]{81\,a^4}$, $\sqrt[12]{8\,b^3}$, $\sqrt[12]{36\,c^2}$. **10.** $\sqrt[4]{81}$, $\sqrt[4]{36}$, $\sqrt[4]{11}$.
11. $\sqrt[mn]{a^{3mn}}$, $\sqrt[mn]{b^{2n}}$, $\sqrt[mn]{c^{4m}}$. **12.** $\sqrt[3]{a^3+3\,a^2b+3\,ab^2+b^3}$, $\sqrt[6]{a^2-2\,ab+b^2}$.
13. $\sqrt[6n]{64\,a^6}$, $\sqrt[6n]{8\,a^3}$, $\sqrt[6n]{4\,a^2}$. **14.** $3\sqrt[6]{125\,a^3x^{-6}}$, $2\sqrt[6]{x^4y^2}$, $4\sqrt[6]{12\,x^3}$.
15. $5\sqrt[12]{100\,x^6}$, $7\sqrt[12]{256\,x^8y^4}$, $6\sqrt[12]{343\,x^9y^6}$. **16.** $\sqrt{7}$. **17.** $\sqrt[4]{3}$.
18. $\sqrt[5]{16}$. **19.** $\sqrt[3]{5}$. **20.** $\sqrt[5]{x^4}$ when $x \geqq 1$, $\sqrt[4]{x^3}$ when $x<1$.
21. $\sqrt[6]{x^5}$ when $x \geqq 1$, $\sqrt[4]{x^3}$ when $x<1$.

Examples LVIII

3. $\frac{4}{3}\sqrt{3}$. **5.** $\frac{7}{10}\sqrt{5\,x}$. **6.** $\frac{1}{2}\sqrt{6}$. **7.** $\frac{6}{49}\sqrt{42}$. **8.** $\frac{8\sqrt{x^2-2}}{x^2-2}$.
9. $\frac{a\sqrt{x^3-y^3}}{b(x-y)}$. **10.** $\frac{a\sqrt{x^3+y^3}}{b(x+y)}$. **11.** $\frac{3}{5}\sqrt{15}$. **12.** $2\sqrt{21}$. **13.** $4\sqrt{55\,x}$.
14. $\frac{2}{3}\sqrt[3]{9}$. **15.** $\frac{2}{3}\sqrt[3]{36}$. **16.** $\frac{4\,a}{7\,bx}\sqrt[3]{16\,x^2}$. **17.** $\frac{1}{3}\sqrt[4]{54}$. **18.** $\frac{3}{4\,x}\sqrt[4]{24\,x^2}$.
19. $\frac{7}{2}\sqrt[5]{48}$. **20.** $3\sqrt[5]{768}$. **21.** $b\sqrt[5]{9\,a^3}$. **22.** $\frac{9}{y}\sqrt[5]{18\,xy}$. **23.** $4\sqrt[4]{4\,x}$.
24. $\frac{1-x^2}{13\,x}\sqrt[3]{13\,x}$. **25.** $(2+x)\sqrt[5]{(2-x)^3}$. **26.** $(x^2-xy+y^2)\sqrt[4]{x+y}$.
27. $(x^2-y^2)\sqrt[5]{(x^2+y^2)^2}$. **29.** $3+\sqrt{2}$. **30.** $\frac{(\sqrt{x}+\sqrt{y})^2}{x-y}$. **31.** $\frac{14+5\sqrt{3}}{11}$.

32. $5(\sqrt{1+x^2}+x)$. **33.** $2x^2-2x\sqrt{x^2+1}+1$. **34.** $x+\sqrt{x^2-1}$.
35. $x+\sqrt{x^2-1}$. **36.** $\sqrt{x^2+a}$. **37.** $x+\sqrt{x^2-1}$. **38.** $\dfrac{\sqrt{a^2+b^2x^2}-a}{bx}$.
40. $2(2+\sqrt{2}-\sqrt{6})$. **41.** $\dfrac{(2\sqrt{2}+\sqrt{3}-\sqrt{5})(2\sqrt{6}-3)}{30}$.

Examples LIX

2. $\sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4}$. **3.** $4-2\sqrt[3]{4}+2\sqrt[3]{2}$.
4. $\sqrt{32}+4\sqrt[6]{5}+2\sqrt{2}\sqrt[3]{5}+2\sqrt{5}+\sqrt{2}\sqrt[3]{25}+\sqrt[6]{5^5}$.
5. $x^3\sqrt[3]{x}+x^2\sqrt[3]{x^2}\sqrt[6]{y^5}+x^2y\sqrt[3]{y^2}+xy^2\sqrt[3]{x}\sqrt{y}+y^3\sqrt[3]{x^2}\sqrt[3]{y}+y^4\sqrt[6]{y}$.

Examples LX

2. $39\sqrt{2}$. **3.** $10\sqrt{3}+21\sqrt{5}$. **4.** $59\sqrt{3}$. **5.** $72\sqrt{7}$. **6.** $8\sqrt{6}$. **7.** $26\sqrt{11}$.
8. $7\sqrt{3}$. **9.** $3\sqrt{5}$. **10.** $15\sqrt[3]{2}$. **11.** $9\sqrt{4}$. **12.** $9ab\sqrt[3]{2ab^2}$.
13. $27x\sqrt[4]{2x}$. **14.** $2x\sqrt[3]{6x^2}$. **15.** $70a^2\sqrt{2}$. **16.** $(9+4x)\sqrt[5]{2x^2}$.
17. $13x(3+2x)\sqrt{2}$. **19.** $16\sqrt{6}$. **20.** $2\sqrt{15}$. **21.** $\frac{7}{12}\sqrt{a}$. **22.** 0.
23. $\sqrt{3}$. **24.** $\sqrt{3}$. **25.** $\frac{9}{20}\sqrt{10}$. **26.** $\frac{4}{7}\sqrt{14}$. **27.** $20\sqrt{3}$. **28.** $4\sqrt[3]{9}$.
29. $17ab\sqrt{2a^2-3b^2}$. **30.** $5ab\sqrt[3]{b+2a}$. **31.** $\dfrac{2a^2}{x^2}$. **32.** $\dfrac{2a}{x}$.
33. $88-4\sqrt{6}$.

Examples LXI

1. 6. **2.** $7\sqrt{3}$. **3.** 10. **4.** 3. **5.** $6x$. **6.** $15\sqrt{30}$. **7.** $72\sqrt{2}$.
8. xy. **9.** 48. **10.** $14\sqrt{6}$. **11.** $30\sqrt{2}$. **12.** $5\sqrt{42}$. **13.** $35\sqrt{91}$.
14. $13\sqrt{21}$. **15.** $77\sqrt{6}$. **16.** $91\sqrt{2}$. **17.** $\frac{1}{3}\sqrt{5}$. **18.** $\frac{1}{2}\sqrt{10}$.
20. $20\sqrt[6]{108}$. **21.** $12\sqrt{3}$. **22.** $2a\sqrt{ab}$. **23.** $\sqrt[5]{27}$. **24.** $3x\sqrt[12]{3x}$.
25. $\sqrt[6]{1960}$. **27.** $\sqrt[4]{135}$. **28.** $3\sqrt[6]{32}$. **29.** $\sqrt[4]{2}$. **30.** $\frac{1}{2}\sqrt[12]{2^{11}}$.
31. $\sqrt[6]{(1+x)^5}$. **32.** $\sqrt[3]{(a^2-b^2)^2}$. **34.** $2+7\sqrt{3}$. **35.** $12x-6+16\sqrt{2x}$.
36. $3+\sqrt{15}+\sqrt{6}$. **37.** 7. **38.** 33. **39.** 3. **40.** a^3. **41.** $4-2\sqrt{10}$.
42. $2+2\sqrt{15}$. **43.** $-48+54\sqrt{6}+12\sqrt{10}+60\sqrt{15}$. **44.** $9a^2+3ab+4b^2$.
45. $18a-x$. **46.** $x+y$. **47.** $3\sqrt[3]{20}-12\sqrt[3]{3}-\sqrt[3]{180}+12$.

Examples LXII

3. 4. **4.** $5\sqrt{3}$. **5.** 2. **6.** $\frac{1}{2}\sqrt{2}$. **7.** $2\sqrt[3]{2a}$. **8.** $2a\sqrt[5]{2}$.
9. $\sqrt{x^2-xy+y^2}$. **10.** $\sqrt{x^2-3y}$. **11.** $\frac{1}{7}\sqrt{35}$. **12.** $\sqrt[6]{15}$. **13.** $\sqrt[3]{36}$.
14. $\dfrac{1}{a}\sqrt[3]{3a^2}$. **15.** $\sqrt[3]{5}$. **16.** $\sqrt[5]{2}$. **17.** $\sqrt[12]{x}$. **18.** $\dfrac{a}{b}\sqrt[6]{3ab}$. **19.** $\frac{1}{2}\sqrt{6}$.

20. 1. **21.** $\sqrt[6]{5}$. **22.** $\sqrt{5}-3\sqrt{3}$. **23.** $8-8\sqrt{3}+\sqrt{5}$. **24.** $\sqrt{7}$.
25. $\sqrt{3}-\frac{1}{3}\sqrt[6]{243}+\frac{1}{3}\sqrt[4]{27}$. **26.** $2+\sqrt{6}$. **27.** $\frac{31}{3}+\frac{8}{3}\sqrt{10}$.
28. $\sqrt{a}+\sqrt{b}+\sqrt{c}$. **29.** $\dfrac{x^4+a^2x^2}{a^4}$. **30.** $\frac{1}{3}\sqrt{6}$.

Examples LXIII

1. $75x$. **2.** $24x$. **3.** $16x\sqrt[3]{25xy^2}$. **4.** $-54ab^2$. **5.** $-\dfrac{8a}{25}\sqrt{5a}$.
6. $9\sqrt{14a}$. **7.** $ab\sqrt{53b}$. **8.** $32a\sqrt[3]{a^2b}$. **9.** $2a^3b^2\sqrt[5]{2a^3b^2}$.
10. $ab\sqrt[3]{7b^2}$. **11.** $5\sqrt{2xy}$. **12.** $-3ab^2$. **13.** $2x\sqrt[3]{y}$.
14. $12-2\sqrt{35}$. **15.** $15+6\sqrt{6}$. **16.** $9-6\sqrt{2}$. **17.** $3\sqrt[3]{18}-3\sqrt[3]{12}-1$.
18. $\sqrt{3}+3\sqrt[6]{36}+3\sqrt[6]{48}+2$.

Examples LXIV

1. $\sqrt[3]{ab}$. **2.** $6\sqrt[3]{5a^2b}$. **3.** $-3\sqrt[3]{4ab^2}$. **4.** $a^2b\sqrt[4]{2ab^2}$. **5.** $2\sqrt[6]{7ax}$.
6. $2\sqrt[5]{xy^2}$. **7.** $\sqrt[3]{a}$. **8.** $a\sqrt[3]{4}$. **9.** $\sqrt[3]{43xy^2}$. **10.** $x(x-y)^n$.
11. $\sqrt[3]{7-5x^2}$. **12.** $2\sqrt{ab}$.

Examples LXV

3. $1+\sqrt{3}$. **4.** $1+\sqrt{2}$. **5.** $\sqrt{5}-\sqrt{2}$. **6.** $\sqrt{6}-\sqrt{2}$. **7.** $5+\sqrt{5}$.
8. $\sqrt{5}-\sqrt{3}$. **10.** $\sqrt{10}+2\sqrt{2}$. **11.** $\sqrt{8}-\sqrt{3}$. **12.** $2+3\sqrt{5}$.

Examples LXVI

4. $8a+15b\sqrt{-1}$. **5.** $78\sqrt{-1}$. **6.** $28\sqrt{-1}$. **7.** $5x\sqrt{-1}$.
8. $2\sqrt{-1}$. **9.** $18\sqrt{3}\sqrt{-1}$. **10.** $14\sqrt{-1}$. **11.** $5a\sqrt{2}\sqrt{-1}$.
12. $10+8\sqrt{-3}$. **13.** $\sqrt[4]{-1}$. **14.** $6+\sqrt[4]{-a}$. **15.** $2\sqrt{-2}$. **16.** $\sqrt{-1}$.

Examples LXVII

3. $-xy$. **4.** $-20\sqrt{15}$. **5.** 180. **6.** -4. **7.** $-42\sqrt{3}$. **8.** -72.
9. $-2(14+\sqrt{2})$. **10.** $4(11-\sqrt{-1})$. **11.** 2. **12.** 153.
14. $15\sqrt[4]{8}\sqrt{-1}\sqrt[4]{-1}$. **15.** $15\sqrt[12]{72}\sqrt{-1}\sqrt[4]{-1}$. **16.** -50. **17.** -24.
18. $256\sqrt[3]{4}$. **19.** $972\sqrt{2}\sqrt{-1}$. **20.** $-128\sqrt{2}\sqrt{-1}$. **21.** 2304.
22, 23. 2. **24, 25.** $\sqrt{31}$. **26, 27.** 5. **28.** $\dfrac{3+\sqrt{-2}}{11}$. **29.** $-\dfrac{7+2\sqrt{6}}{5}$.
30. $-\frac{2}{3}\sqrt{-1}$. **31.** $-19-6\sqrt{10}$. **34.** 12. **35.** $\frac{2}{3}\sqrt{3}$. **36.** $-\sqrt{-1}$.
37. $-3\sqrt{-1}$. **38.** $7\sqrt[3]{4}$. **39.** $\sqrt[6]{2}\sqrt[4]{-1}$. **41.** $\sqrt{-1}$. **42.** $\dfrac{3+2\sqrt{-3}}{21}$.

43. $\frac{a^2 - x + 2a\sqrt{-x}}{a^2 + x}$. **44.** 1. **45.** $\frac{2(a^2 - b)}{a^2 + b^2}$. **47.** $2 + 3\sqrt{-1}$.

48. $3 - 4\sqrt{-1}$. **49.** $4 + 3\sqrt{-1}$. **50.** $5 - 2\sqrt{-1}$.

52. $6 - \sqrt{-1}$. **54.** $7 - 4\sqrt{-1}$. **55.** $6 + 3\sqrt{-1}$.

57. $-\frac{8}{29}$. **8.** $\frac{6 + 2\sqrt{5x} - 2\sqrt{2x - 6}}{5 - x}$.

Examples LXVIII

3. -10. **4.** -55. **5.** $23\frac{1}{4}$. **6.** $\frac{4}{3}$. **7.** 7. **8.** 4. **9.** 4.

10. 1. **11.** -1. **12.** 4. **13.** $\frac{b}{c}$. **14.** $\frac{1}{3}$. **15.** $4\frac{1}{2}$. **16.** 2.

17. 3. **18.** $\frac{ac}{b}$. **19.** 0. **20.** $-a$. **21.** 0. **22.** $-\frac{2}{7}$.

Examples LXIX

1. 81. **2.** $c^2 - 2bc$. **3.** 16. **4.** 5. **5.** $4(a - 1)$. **6.** 8.

7. 1. **8.** 6. **9.** 6. **10.** 3. **11.** $\pm\frac{2a\sqrt{b}}{b + 1}$. **12.** $\frac{81}{a}$. **13.** 4.

14. $\left(\frac{2b + \sqrt{ab}}{a}\right)^2$. **15.** $a\left\{1 - \left(\frac{2\sqrt{b}}{b + 1}\right)^4\right\}$. **16.** $\frac{4m^2}{(m + 1)^2}$.

17. $\frac{a}{2b}(b - 1)^2$. **18.** $\frac{9}{7}$. **19.** $\sqrt{a^2 - \left(\frac{b^3 - 2a}{3b}\right)^3}$. **20.** $\frac{(a - 1)^2}{2a}$.

21. $\frac{3}{16}$. **22.** $\frac{3a}{4}$. **23.** 1. **24.** 4.

Examples LXX

3. 35. **4.** $\frac{anq}{nq - mq - np}$. **5.** $112\frac{1}{2}$. **6.** $\frac{an + bm}{an + bn}$. **7.** $\frac{11}{14}$. **8.** 175.

9. $\frac{8}{11}$. **10.** 40. **11.** 30. **12.** $\frac{105}{136}$. **13.** 6. **14.** 77. **15.** 40.

16. 1. **7.** \$1200. **18.** $\frac{n(c - b)}{a - b}$, $\frac{n(a - c)}{a - b}$. **19.** 40, 80.

20. 6, 15, ∞. **21.** 5712. **22.** 105. **23.** 50. **24.** A, 32; B, 25.

26. $\begin{cases} 3 \text{ hr. } 16\frac{4}{11} \text{ m.} \\ 3 \text{ hr. } 32\frac{8}{11} \text{ m.} \end{cases}$ **27.** 4 hr. $41\frac{5}{11}$ m. **28.** 4 hr., 54 mi., 60 mi.

29. 43. **30.** 142,857. **31.** $\frac{am}{m + n}$. **32.** \$1. **33.** 30. **34.** 300.

35. 15, 90. **36.** 10. **37.** 8. **38.** 10. **39.** 140, 60, 45, 80.

40. 50, \$1350, \$1200. **41.** 45, 30.

Examples LXXI

1. 10, 7. **2.** 10, 3. **3.** 4, 1. **4.** 2, 2. **5.** 2, -3. **6.** 12, 9.

7. 1, 2. **8.** -3, 5. **9.** -2, -4. **10.** -2, 1. **11.** 6, -10.

12. $4, -3$. **13.** $1, -2$. **14.** $-8, 5$. **15.** $18, 12$. **16.** $-2, -6$.

17. $\dfrac{bc'-b'c}{a'b-ab'}, \dfrac{a'c-ac'}{a'b-ab'}$. **18.** $\dfrac{3b}{2}, -\dfrac{a}{2}$. **19.** a, a. **20.** $\dfrac{a+2b}{2}, \dfrac{a-2b}{2}$.

21. $4, -6$. **22.** $-6, -2$. **23.** $\dfrac{1}{a}, \dfrac{1}{b}$. **24.** $\dfrac{a}{b}, -\dfrac{b}{a}$. **25.** $\dfrac{1}{b}, \dfrac{1}{a}$.

26. $a+b, \dfrac{1}{a+b}$.

Examples LXXII

4. $18\frac{3}{4}, 31\frac{1}{4}$. **5.** $5, 60$. **6.** $30, 90$. **7.** $\frac{10}{9}$. **8.** $3, 9$. **9.** 24.

10. $15, 22\frac{1}{2}$. **11.** $24, 30$. **12.** $\dfrac{bm+a}{mn-1}, \dfrac{an+b}{mn-1}$. **13.** $48, 16$.

14. $30, 20$. **15.** $42, 11, 7$. **16.** $50, 75$. **17.** $4\frac{1}{2}, 7\frac{1}{2}$.

18. $\dfrac{pm+qn-qmn}{mn-m-n}, \dfrac{pmn-qn-pm}{mn-m-n}$. **19.** $30, 20$. **20.** $20, 8$.

21. $10\frac{1}{5}, 17$. **22.** $24, 32$. **23.** $16, 24$. **24.** $\$10,000, \$7200$.

25. $100, 12$. **26.** $\$3000, 4\%$. **27.** $11\frac{1}{7}, 9\frac{3}{4}$. **28.** $\dfrac{a(b-c)}{b-a}, \dfrac{b(c-a)}{b-a}$.

29. $50, 30$. **30.** $18, 15$. **31.** $48, 80$; $\$63, \$33$. **33.** $5\%, 8\%$.

34. $\dfrac{n(r-p)}{r-q}, \dfrac{n(p-q)}{r-q}$. **35.** $11, 7$.

Examples LXXIII

1. $1, 2, 3$. **2.** $2, 3, 4$. **3.** $3, 2, 1$. **4.** $2, 5, -1$. **5.** $\frac{11}{16}, -\frac{7}{8}, \frac{1}{4}$.

6. $25, 55, 65$. **7.** $3, 4, 7, 1$. **8.** $6, 3, 4, 5$. **9.** $-5, 4, -3, -2$.

10. $2, 1, 1, 2$. **11.** $\frac{2}{3}, 2, 1$. **12.** $\frac{7}{6}, -\frac{7}{2}, \frac{21}{10}$. **13.** $3\frac{3}{7}, 2\frac{2}{3}, 2\frac{2}{11}$.

14. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. **15.** $2a, 2b, 2c$. **16.** $\frac{5}{4}, \frac{3}{2}, \frac{4}{3}$. **17.** $\frac{2}{5}, \frac{2}{3}, 2$. **18.** $1, 2, 3$.

19. $3\frac{3}{7}, 2\frac{2}{5}, 2\frac{2}{3}$. **20.** $-\dfrac{2bc}{b+c}, -\dfrac{2ac}{a+c}, -\dfrac{2ab}{a+b}$. **21.** $4, 5, 6$.

22. $2, 1, 4, 3$. **23.** $1, -2, 4, -3$. **24.** $3, 2, -5, 1$. **25.** $1, 2, 3, 4, 5$.

26. $2, 1, -2, 3, -1, 4$. **27.** $2, 1, 3, 4$. **28.** $4, -2, 1, -3$.

29. $1, 3, 5, 7$. **30.** $3, 4, -6, 5$. **31.** $4, 6, 8, 10, 12$. **32.** $5, -3, 7, -2, 4$.

Examples LXXIV

1. 426. **2.** \$2, 20 cts., 10 cts. **3.** 37, 45, 52. **4.** 480, 400, 560.

5. \$760, 200, 200. **6.** \$1000, \$– 500, \$0. **7.** 120, 114, 110.

8. 1 to 2, 6 min. **9.** 21, 17, 23, 15. **10.** $18\frac{6}{13}, 34\frac{2}{7}, 23\frac{7}{31}$, 80.

Examples LXXV

4. The 1st. **7.** Less. **9.** The 1st. **11.** The 1st. **12.** Between 34 and 37. **13.** Between 15 and 20. **14.** Between \$12 and \$14. **15.** 15, 12.

16. $7\frac{1}{2}$. **17.** 5. **18.** 20 or 5.

Examples LXXVI

1. $x-5$. **2.** $\frac{1}{x-2}$. **3.** $x+y$. **4.** $\frac{3(a-b)}{2(a+b)}$. **5.** $\frac{x-3}{x-1}$.
6. $\frac{1}{9}$. **7.** 11. **8.** $\frac{64}{125}$. **9.** $\frac{3}{4}$. **10.** Greater. **11.** Greater.
12. Less. **13.** 15, 21. **14.** 14, 35. **15.** 7. **16.** 5 : 8.
17. $\frac{m^2n^2}{m^2+mn+n^2}$. **18.** 4 : 1. **19.** 3 : 2. **20.** 120, 160, 200.
21. 42, 12. **22.** 8 : 9. **23.** 3 : 4. **24.** 242, 300, 358. **25.** 16, 20.
26. B, 11 : 10. **28.** (*a*) 72, (*b*) 84, (*c*) 75, (*d*) 77.

Examples LXXVII

1. $\frac{5}{6}$. **3.** $\frac{4}{7}$. **4.** $xy-\frac{1}{xy}$. **5.** 3. **6.** 3, 12. **13.** $\frac{bc-ad}{a-b-c+d}$.
18. $\frac{b^2-ac}{2b-a-c}$. **20.** 200, 300. **21.** 8, 6. **22.** 2 or $10\frac{1}{2}$. **23.** 13, 26, or 39.
24. 45, 54. **25.** $\frac{2p}{s-d}$, $\frac{2p}{s+d}$. **26.** $\frac{n}{2m}+\frac{b}{2a}$, $\frac{n}{2m}-\frac{b}{2a}$. **27.** 78, 66.
28. .0089. **29.** $\frac{r^2v}{r'^2}$. **31.** 8 : 45 A.M. **32.** 10. **33.** 6. **34.** 45, 30.
35. 100.

Examples LXXVIII

2. $x \propto y$. **3.** $x \propto z^4$. **6.** $x=8y$. **7.** $x=\frac{52}{y}$. **8.** $x=\frac{2}{y^2}$. **10.** 24.
11. 5. **12.** 9. **13.** $\frac{1}{2}$. **14.** 4. **15.** 6. **17.** 48. **18.** 579.
19. $67\frac{1}{2}$. **20.** 18. **21.** 47. **22.** 250. **23.** $26\frac{2}{3}$. **25.** $y=\frac{4}{3}x+\frac{14}{3x}$.
26. $x=1+2y+3y^2$. **27.** $s=\frac{1}{2}ft^2$. **28.** $r\left(\frac{t'}{t}\right)^{\frac{2}{3}}$. **29.** 27, 74.
30. .00611.

Examples LXXIX

1. 58, 590. **2.** 83, 903. **3.** -58, -828. **4.** -3, 0. **5.** $l=116$, $S=1404$. **6.** $a=1$, $S=540$. **7.** $d=6$, $S=473$. **8.** $n=12$, $S=582$.
9. $a=32$, $l=84$. **10.** $d=-5$, $l=-95$. **11.** $n=18$, $d=8$.
12. $a=\frac{3}{5}$, $d=-\frac{1}{15}$. **13.** 103, 133, 163. **14.** $d=9$. **15.** -46.
16. $\frac{2}{5}$. **17.** 576. **18.** 1512. **19.** 78th and 79th. **20.** $(2t-1)a$, at^2.
21. 14,475. **22.** 10 sec., $4\frac{2}{3}$ min. **23.** 19,600 ft. **24.** 11.

Examples LXXX

2. 2048, 4095. **3.** 19,683, 29,524. **4.** 16,384, $21,845\frac{21}{64}$. **5.** 3072, 2049.
6. $a=5$, $l=640$. **7.** $a=4$, $S=39,364$. **8.** $l=-\frac{1}{96}$, $S=-\frac{43}{96}$.
9. $a=1$, $S=511$. **10.** $r=-4$, $S=1638$. **11.** $r=\pm 3$, $S=2186$ or -1094.
12. $n=5$, $S=121$. **13.** $a=5$, $n=6$. **14.** 68, 272, 1088. **15.** $r=3$.
16. 21. **17.** 13,120. **18.** 3072. **19.** ± 2. **20.** 2. **21.** 6, 24, 96, 384, 1536.

22. 2. **23.** $\frac{3}{2}$. **24.** $\frac{3}{4}$. **25.** $\frac{419}{990}$. **26.** $\frac{5}{18}$. **27.** $\frac{103}{330}$. **28.** $\frac{1}{2}$.
29. 150. **30.** (*a*) \$127,093,291,416.30; (*b*) \$25,418,658,283.26. (*c*) \$169,457.72.

Examples LXXXI

1. $\frac{1}{113}$. **2.** $\frac{1}{13}$, $\frac{1}{21}$. **3.** 28. **4.** $16\frac{4}{5}$, 21, 28. **5.** $\frac{1}{29}$, $\frac{1}{35}$, $\frac{1}{41}$.

Examples LXXXII

1. ± 3. **2.** ± 5. **3.** ± 1. **4.** ± 1. **5.** ± 3. **6.** $\pm\sqrt{19}$
7. $\pm\sqrt{\dfrac{b}{a-1}}$. **8.** $\pm\frac{1}{2}$. **9.** $\pm\sqrt{2ab-b^2}$. **10.** $\pm\dfrac{a}{3}\sqrt{3}$. **11.** $\pm\sqrt{3}$.
12. $\pm\dfrac{2}{\sqrt{4ab-b^2}}$.

Examples LXXXIII

1. 6, 15. **2.** $\dfrac{m\sqrt{s}}{\sqrt{m^2+n^2+p^2}}$, $\dfrac{n\sqrt{s}}{\sqrt{m^2+n^2+p^2}}$, $\dfrac{p\sqrt{s}}{\sqrt{m^2+n^2+p^2}}$. **3.** 6, 15.
4. 12 ft., 18 ft. **5.** 4, 16. **6.** 40, 36. **7.** 4550. **8.** 3. **9.** 5.57+, 18.12−.
10. 161124.3+. **11.** $\dfrac{132\sqrt{7}}{\sqrt{7}\pm\sqrt{17}}$ from the weaker light.
12. $\dfrac{a\sqrt{3}}{1\pm\sqrt{3}}$ from the weaker bell. **13.** 5. **14.** $\dfrac{n}{\sqrt{2n-m}}$, $\dfrac{m-n}{\sqrt{2n-m}}$.
15. 20, 10. **16.** 150. **17.** 48.

Examples LXXXIV

3. 5, 1. **4.** −1, −5. **5.** 11, −1. **6.** −1, −9. **7.** 11, 1. **8.** 8, −6.
9. 8, −2. **10.** 6, 2. **11.** 10, −6. **12.** 8, −1. **13.** 32, −2.
14. 14, −2. **15.** 15, −3. **16.** 11, −3. **17.** 4, $-\frac{1}{2}$. **18.** 2, −1.
19. $\dfrac{b^2}{ac}$, $\dfrac{b^2}{ac}$. **20.** 2, $-5\frac{1}{3}$. **21.** $\dfrac{b}{a}$, $-\dfrac{a}{b}$. **22.** $\dfrac{a\pm b}{c}$. **23.** $\dfrac{3b}{4}$, $\dfrac{b}{2}$.

Examples LXXXV

3. 12, 36. **4.** 14, 10. **5.** 6. **6.** 20, 3. **7.** 12. **8.** 70, 80. **9.** 40.
10. 7. **11.** 10, $10\frac{1}{2}$. **12.** 3. **13.** 18, \$20. **14.** 15. **15.** 576. **16.** 12 or 8.
17. 9, 36. **18.** 1 hr. 48 min., 2 hr. 15 min. **19.** 3 or $4\frac{1}{5}$. **20.** 15.

Examples LXXXVI

3. 5, 1. **4.** 8, −3. **5.** 3, 3. **6.** 7, −7. **7.** 0, 7. **8.** 6, 6.
9. $x^2-2x=8$. **10.** $x^2-10x=-21$. **11.** $x^2+3x=40$.
12. $x^2-15x=-54$. **13.** $x^2=3$. **14.** $x^2-4x=1$. **15.** 6, 4.
16. 6, 2. **17.** $2p^2=9q$. **18.** $q=-1$.

Examples LXXXVII

1. 3, 0. **2.** 5. (Reject -2). **3.** 6, 7. **4.** 1, 8, -5. **5.** -1, ± 4.
6. 1, $\frac{1}{81}$. **7.** 0, $4(a+b)$. **8.** 3, 2. **9.** 4, $\frac{1}{4}$. **10.** 8, $-\frac{8}{9}$. **11.** 4, $\frac{1}{9}$.
12. $\pm\sqrt{3}$. **13.** 5, 3. **14.** $\pm\frac{1}{2}\sqrt{3}$. **15.** 0, 4, 12. **16.** ± 5.

Examples LXXXVIII

1. ± 4. **2.** 3. **3.** 36. **4.** ± 729. **5.** ± 125. **6.** 27. **7.** $\frac{7}{3}$. **8.** $\frac{9}{4}$. **9.** 64.

Examples LXXXIX

2. 2. **3.** 7, -1. **4.** ± 5. **5.** ± 2. **6.** -3. **7.** ± 2.
8. $\frac{1}{2}(5\pm\sqrt{5})$. **9.** $-\frac{1}{4}(1\pm\sqrt{13})$. **10.** 1, -5.

Examples XC

2. ± 2, ± 1. **3.** $\pm\sqrt{5}$, ± 1. **4.** ± 3, $\pm\sqrt{-1}$. **5.** ± 3, ± 4.
6. $\sqrt[3]{2}$, -2. **7.** 4, $\sqrt[3]{49}$. **8.** 16, $\frac{14641}{625}$. **9.** 243, $(-28)^{\frac{5}{3}}$. **10.** 8, $\frac{125}{64}$.
11. $\pm\frac{1}{6}\sqrt{30}$, ± 1. **12.** 64, $(-33)^{\frac{6}{5}}$. **13.** $\dfrac{1}{\sqrt[n]{3}}$, $\dfrac{1}{\sqrt[n]{-6}}$. **14.** $\sqrt[2n]{8}$, $\sqrt[2n]{-\frac{512}{27}}$.
15. 16, $\frac{1}{16}$. **16.** 2, $\frac{1}{2}$.

Examples XCI

2. 5, $-2\frac{1}{2}$. **3.** 1, 1. **4.** 2, $-6\frac{8}{9}$. **5.** 9, -12. **6.** ± 5, $\pm 3\sqrt{2}$.
7. 3, $-\frac{1}{2}$. **8.** 1, 0, -3, -4. **9.** 4, 69. **10.** 8, -1, $\frac{1}{2}(7\pm\sqrt{53})$.
11. 2, $\frac{3}{2}$, $\frac{1}{4}(7\pm\sqrt{33})$. **12.** 2, $-\frac{1}{2}$, $\frac{1}{4}(3\pm\sqrt{505})$. **13.** 3, -1, $1\pm\frac{1}{2}\sqrt{61}$.
14. 2, $-\frac{1}{2}$, $\frac{1}{2}(-17\pm\sqrt{305})$. **15.** 1, $\frac{1}{2}(-3\pm\sqrt{5})$. **16.** 5, -2.
17. 3, $-3\frac{1}{3}$, $\frac{1}{6}(-1\pm\sqrt{-251})$. **18.** $\frac{1}{2}(1\pm\sqrt{5})$. **19.** $\dfrac{a}{2}(1\pm\sqrt{5})$.

Examples XCIV

1. $\begin{cases} 3, -\frac{53}{27}; \\ -4, \frac{227}{27}. \end{cases}$ **2.** $\begin{cases} \pm\sqrt{\frac{5}{2}}; \\ 2\mp\sqrt{\frac{5}{2}}. \end{cases}$ **3.** $\begin{cases} 2; \\ 2. \end{cases}$ **4.** $\begin{cases} 1, -1\frac{8}{9}; \\ 2, -3\frac{7}{9}. \end{cases}$ **5.** $\begin{cases} 5, 10; \\ 10, 5. \end{cases}$ **6.** $\begin{cases} a, a; \\ b, b. \end{cases}$

Examples XCV

1. $\begin{cases} 2, -5; \\ 4, -3. \end{cases}$ **2.** $\begin{cases} 1, -\frac{7}{3}; \\ 0, -\frac{5}{3}. \end{cases}$ **3.** $\begin{cases} -2, 0; \\ 2, 1. \end{cases}$ **4.** $\begin{cases} \frac{1}{2}\pm\frac{1}{6}\sqrt{33}; \\ -1\pm\frac{1}{3}\sqrt{33}. \end{cases}$

Examples XCVI

1. $\begin{cases} 1, -1\frac{1}{5}, 3, -2; \\ 3, -3\frac{3}{5}, -3, 2. \end{cases}$ **2.** $\begin{cases} \pm 6; \\ \pm 3. \end{cases}$

3. $\begin{cases} 3, -2\frac{1}{6}, \frac{1}{4}(1\pm\sqrt{105}); \\ 1, -1\frac{3}{18}, \frac{1}{4}(1\pm\sqrt{105}). \end{cases}$ **4.** $\begin{cases} 1, -1\frac{1}{5}, \frac{1}{34}(-1\pm\sqrt{409}); \\ 2, -2\frac{2}{5}, \frac{2}{17}(-1\pm\sqrt{409}). \end{cases}$

Examples XCVII

1. $\begin{cases} \pm 3, \pm \frac{5}{2}\sqrt{2}; \\ \pm 2, \pm \frac{1}{2}\sqrt{2}. \end{cases}$ 2. $\begin{cases} \pm 2, \pm \frac{1}{5}\sqrt{10}; \\ \pm \frac{1}{2}, \mp \frac{2}{5}\sqrt{10}. \end{cases}$ 3. $\begin{cases} \pm 3, \mp 8; \\ \pm 5. \end{cases}$

4. $\begin{cases} \pm 1, \pm \frac{11}{5}\sqrt{-5}; \\ \pm 2, \pm \frac{4}{5}\sqrt{-5}. \end{cases}$ 5. $\begin{cases} \pm 2, \pm \frac{4}{3}\sqrt{3}; \\ \pm 6, \pm \frac{10}{3}\sqrt{3}. \end{cases}$ 6. $\begin{cases} \pm 10, \pm \frac{14}{47}\sqrt{-47}; \\ \pm 3, \mp \frac{57}{47}\sqrt{-47}. \end{cases}$

Examples XCVIII

1. $\begin{cases} \pm 4; \\ \pm 2. \end{cases}$ 2. $\begin{cases} 7, -5. \\ 5, -7. \end{cases}$ 3. $\begin{cases} 3, 2; \\ 2, 3. \end{cases}$ 4. $\begin{cases} 6, -5; \\ -5, 6. \end{cases}$

5. $\begin{cases} \pm \frac{5}{7}\sqrt{21}; \\ \pm \frac{1}{7}\sqrt{21}. \end{cases}$ 6. $\begin{cases} 2, 10, -2, -10; \\ 1, -3, -1, 3. \end{cases}$ 7. $\begin{cases} 10, -5; \\ 5, -10. \end{cases}$ 8. $\begin{cases} \pm 3, \pm 2; \\ \pm 2, \pm 3. \end{cases}$

Examples XCIX

2. $\begin{cases} 4, 2, \frac{1}{6}(-13 \pm \sqrt{377}); \\ 2, 4, \frac{1}{6}(-13 \mp \sqrt{377}). \end{cases}$ 3. $\begin{cases} 3, 1, 2 \pm \sqrt{\frac{113}{7}}; \\ 1, 3, 2 \mp \sqrt{\frac{113}{7}}. \end{cases}$

4. $\begin{cases} 0, 12, 6; \\ 0, 6, 12. \end{cases}$ 5. $\begin{cases} 4, 2, \frac{1}{2}(-7 \pm \sqrt{-35}); \\ 2, 4, \frac{1}{2}(-7 \mp \sqrt{-35}). \end{cases}$

Examples C

1. $\begin{cases} \pm \frac{6}{7}\sqrt{14}; \\ \pm \frac{1}{7}\sqrt{14}. \end{cases}$ 2. $\begin{cases} \frac{1}{2}(3 \pm \sqrt{13}); \\ \frac{1}{2}(1 \pm \sqrt{13}). \end{cases}$ 3. $\begin{cases} 9; \\ 6. \end{cases}$ 4. $\begin{cases} \pm 4, \pm 14; \\ \pm 1, \mp 4. \end{cases}$

5. $\begin{cases} 3, -1, -\frac{7}{23}(-3 \pm 2\sqrt{-15}); \\ 3, -1, \frac{1}{23}(-3 \pm 2\sqrt{-15}). \end{cases}$ 6. $\begin{cases} \pm 8, \pm 2\sqrt{-1}; \\ \pm 2, \mp 8\sqrt{-1}. \end{cases}$ 7. $\begin{cases} 3, 4; \\ 4, 3. \end{cases}$

8. $\begin{cases} \pm \frac{4}{3}\sqrt{6}, \pm 3; \\ \pm \frac{1}{6}\sqrt{6}, \pm 1. \end{cases}$ 9. $\begin{cases} 2, 3; \\ 3, 2. \end{cases}$ 10. $\begin{cases} 3, 2, \frac{1}{2}(5 \pm \sqrt{-151}); \\ 2, 3, \frac{1}{2}(5 \mp \sqrt{-151}). \end{cases}$ 11. $\begin{cases} 3, 1; \\ 1, 3. \end{cases}$

12. $\begin{cases} \pm a, \pm \dfrac{a}{3}; \\ \pm \dfrac{b}{3}, \pm b. \end{cases}$ 13. $\begin{cases} \pm 3, \pm \frac{5}{2}\sqrt{2}; \\ \pm 2, \pm \frac{1}{2}\sqrt{2}. \end{cases}$ 14. $\begin{cases} \pm 9, \pm 4; \\ \pm 4, \pm 9. \end{cases}$ 15. $\begin{cases} 3, -1; \\ 1, -3. \end{cases}$

16. $\begin{cases} 5, -4\frac{1}{2}; \\ 3, -3\frac{1}{3}. \end{cases}$ 17. $\begin{cases} 0, -4, 0, \frac{12}{7}; \\ 0, 2, 0, \frac{4}{7}. \end{cases}$ 18. $\begin{cases} \pm 5; \\ \pm 3. \end{cases}$ 19. $\begin{cases} 5, 3; \\ 3, 5. \end{cases}$ 20. $\begin{cases} 1, 2 \pm \sqrt{7}; \\ 1, 2 \mp \sqrt{7}. \end{cases}$

Examples CI

1. 20, 15. **2.** 20, 15. **3.** 10, \$125. **4.** 4, 5. **5.** Dis., 15; rates, 3, $2\frac{1}{2}$, 4. **6.** 48, 42. **7.** 6, $1\frac{1}{2}$. **8.** Dis., $46\frac{2}{3}$ or 30; rate, 4 or 3. **9.** Rowing, $4\frac{1}{2}$ and 6; walking, $4\frac{1}{2}$. **10.** 16, $5\frac{1}{3}$.

Examples CII

3. $x = 3$ renders $f(x)$ a max. $f(x)$ at a max. $= 9$.

4. $x = 4$ renders $f(x)$ a min. $f(x)$ at a min. $= 3$.

5. $x = -2$ renders $f(x)$ a min. $f(x)$ at a min. $= 0$.
6. $x = 2$ renders $f(x)$ a max. $f(x)$ at a max. $= 15$.
7. $x = 2$ renders $f(x)$ a min. $f(x)$ at a min. $= 0$.
8. $x = 4$ renders $f(x)$ a max. $f(x)$ at a max. $= 50$.
9. $x = 0$ renders $f(x)$ a min. $f(x)$ at a min. $= 11$.
10. $x = -\frac{1}{2}$ renders $f(x)$ a max. $f(x)$ at a max. $= 7\frac{3}{4}$.

Examples CIII

2. $x = -1$ renders $f(x)$ a min. $f(x)$ at a min. $= \frac{1}{7}$.
3. $x = -\frac{3}{2}$ renders $f(x)$ a max. $f(x)$ at a max. $= 6$.
4. $x = 4$ renders $f(x)$ a max. $f(x)$ at a max. $= -2m$.
5. $x = 2$ renders $f(x)$ a min. $f(x)$ at a min. $= \dfrac{n-4}{23}$.

Examples CIV

8. $f(x)$ at a min. $= -\frac{1}{6}$, when $x = 3$. $f(x)$ at a max. $= -\frac{1}{2}$, when $x = 1$.
9. $f(x)$ at a min. $= b$, when $x = a$.
10. $f(x)$ at a max. $= \frac{1}{3}$, when $x = 3$.
11. $f(x)$ at a max. $= \frac{1}{5}$, when $x = -5$.
12. $f(x)$ at a min. $= 0$, when $x = -1$.
13. $f(x)$ at a max. $= a$, when $x = a$.
14. $f(x)$ at a min. $= 6$, when $x = \pm 2$. $f(x)$ at a max. $= 10$, when $x = 0$.
15. $f(x)$ at a min. $= -\frac{1}{3}$, when $x = 2$. $f(x)$ at a max. $= 1$, when $x = 0$.
16. $f(x)$ at a max. $= b$, when $x = a$.
17. $f(x)$ at a min. $= -27$, when $x = \pm\sqrt{2}$. $f(x)$ at a max. $= -15$, when $x = 0$.

Examples CV

2. $\left\{\begin{array}{l} y \text{ at a max.} = 6, \text{ when } x = 1. \quad y \text{ at a min.} = -2, \text{ when } x = 1. \\ x \text{ has neither a max. nor a min.} \end{array}\right.$
3. $\left\{\begin{array}{ll} y \text{ at a max.} = 4, \text{ when } x = 2. & y \text{ at a min.} = -2, \text{ when } x = 2. \\ x \text{ at a max.} = 2 + \sqrt{22}, \text{ when } y = 1. & x \text{ at a min.} = 2 - \sqrt{22}, \text{ when } y = 1. \end{array}\right.$
4. $\left\{\begin{array}{ll} y \text{ at a max.} = 5, \text{ when } x = 0. & y \text{ at a min.} = -3, \text{ when } x = 0. \\ x \text{ at a max.} = 4, \text{ when } y = 1. & x \text{ at a min.} = -4, \text{ when } y = 1. \end{array}\right.$

Examples CVI

3. 6 and 6. Min. val. $= 72$. 4. A square, each side 40 rd. 5. 384 sq. yd.
7. 6 knots, 32 m. 8. Dimensions, 8 by 16. Area, 128 sq. rd.
9. The corners are at the middle points of the given square. Side of min. square $= \dfrac{a}{\sqrt{2}}$. 11. The length is twice the breadth.
12. Each side is $\frac{1}{2}\sqrt{2}$ times the hypotenuse. 13. 7 ft. by $1\frac{3}{4}$ ft.

Examples CVII

3. ∞. **4.** ∞. **5.** 0. **6.** 0. **7.** 3. **8.** $\frac{4}{5}$. **9.** $\frac{3}{5}$. **10.** $\frac{c}{f}$. **11.** $\frac{2}{3}$. **12.** 1.

Examples CIX

2. $dy = (3x^2 + 10x - 6)dx.$

3. $dy = -(3x^{-4} + 8x)dx.$

4. $dy = (28x^3 + 6x - 3x^{-\frac{1}{2}})dx.$

5. $dy = 10(x^4 + 2x^{-5} - x)dx.$

7. $dy = 120(2 + 3x^2)^3 x dx.$

8. $dy = -120(1 - 5x^2)^2 x dx.$

9. $dy = 5b(a + bx^3)^{\frac{2}{3}} x^2 dx.$

10. $dy = 12(3 - x^3)^{-3} x^2 dx.$

12. $du = 3x^2 y^{\frac{3}{2}} dx + \frac{3}{2} x^3 y^{\frac{1}{2}} dy.$

13. $du = (y^2 - 3)dx + 2xy dy.$

14. $du = (y^3 + 2)dx + 3(x - 1)y^2 dy.$

15. $(x + a)^{\frac{5}{2}} dx + \frac{5}{2} x(x + a)^{\frac{3}{2}} dx.$

17. $dy = -\dfrac{3\,dx}{(1 + x)^4}.$

18. $du = \dfrac{3x^2 y dx - 2x^3 dy}{y^3}.$

20. $dy = \dfrac{dx}{2\sqrt{1 + x}}.$

21. $dy = \dfrac{4(1 - 3x)}{\sqrt{2x - 3x^2}} dx.$

Examples CX

2. $\dfrac{dy}{dx} = 3x^2 - 6x.$

3. $\dfrac{dy}{dx} = 9(x^3 - 5)^2 x^2.$

4. $f'(x) = \dfrac{x}{\sqrt{x^2 - 2}}.$

5. $f'(x) = 2x + 1 - 3x^{-4}.$

6. $\dfrac{dy}{dx} = -\dfrac{2m}{(1 + x)^3}.$

7. $f'(x) = -\dfrac{4m}{(x + a)^5}.$

9. $\dfrac{d^2y}{dx^2} = 2.$

10. $\dfrac{d^2y}{dx^2} = 36x^2 - 10.$

11. $f''(x) = 450(2 + 5x).$

12. $12(a + x)^{-5}.$

13. $\dfrac{d^2y}{dx^2} = \dfrac{2m}{(a + x)^3}.$

14. $80(2x^2 - 3)^2(14x^2 - 3).$

Examples CXI

1. $1 + 3x + 4x^2 + 7x^3 + 11x^4 +$ etc.

2. $2 - 5x + 3x^2 + 2x^3 - 5x^4 +$ etc.

3. $1 + 2x + x^2 - x^3 - 2x^4 - x^5 +$ etc.

4. $1 - x - x^2 + 5x^3 - 7x^4 - x^5 +$ etc.

5. $5 + 27x + 130x^2 + 623x^3 +$ etc.

6. $1 + 2x + 3x^2 + 6x^3 + 9x^4 +$ etc.

7. $2 + 2x - 3x^2 - 5x^3 - x^4 -$ etc.

8. $1 + x + x^2 - x^3 +$ etc.

9. $\frac{1}{2} + \frac{5}{4}x + \frac{7}{8}x^2 + \frac{17}{16}x^3 +$ etc.

Examples CXII

1. $\frac{2}{3}x^{-2} + \frac{8}{9}x^{-1} + \frac{32}{27} + \frac{128}{81}x + \frac{512}{243}x^2 +$ etc.

2. $x^{-1} + 3 + 2x - 5x^2 - 16x^3 +$ etc.

3. $\frac{2}{3}x + \frac{4}{9}x^3 + \frac{8}{27}x^5 + \frac{16}{81}x^7 +$ etc.

4. $x^{-2} - x^{-1} - 2x + 2x^2 - 4x^3 +$ etc.

5. $x - x^2 - 2x^3 - 5x^4 - 12x^5 -$ etc.

6. $\frac{3}{2}x^{-3} - \frac{1}{4}x^{-2} - \frac{1}{8}x^{-1} + \frac{31}{16} + \frac{23}{32}x +$ etc.

Examples CXIII

2. $1 + \frac{1}{2}x + \frac{3}{8}x^2 - \frac{3}{16}x^3 + \frac{3}{128}x^4 +$ etc.

3. $1 + \frac{1}{2}x - \frac{5}{8}x^2 + \frac{5}{16}x^3 +$ etc.

4. $3 + \frac{1}{6}x - \frac{109}{216}x^2 + \frac{109}{3888}x^3 +$ etc.

6. $1 + \frac{1}{3}x + \frac{2}{9}x^2 - \frac{13}{81}x^3 + \frac{8}{243}x^4 +$ etc.

Examples CXIV

3. $-\frac{2}{x}-\frac{1}{2(1-x)}+\frac{1}{2(1+x)}$.

4. $\frac{3}{2(x-2)}-\frac{1}{2x}$.

5. $\frac{5}{3(x-2)}-\frac{2}{3(x+1)}$.

6. $\frac{5}{x-4}-\frac{4}{x-3}$.

7. $\frac{2}{(x-3)^3}-\frac{5}{(x-3)^2}+\frac{1}{x-3}$.

8. $\frac{2}{x+5}-\frac{23}{(x+5)^2}$.

9. $\frac{3}{x+1}-\frac{6}{(x+1)^2}-\frac{1}{(x+1)^3}$.

10. $\frac{1}{5(5x-2)}-\frac{4}{5(5x-2)^3}$.

11. $-\frac{2}{x}+\frac{3}{3x+1}+\frac{2}{2x-5}$.

12. $\frac{1}{2(x+1)}-\frac{4}{x+2}+\frac{9}{2(x+3)}$.

13. $-\frac{2}{x}+\frac{3}{x+2}+\frac{5}{(x+2)^2}$.

14. $\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{(x-2)^2}$.

Examples CXV

3. $\frac{2x-11}{x^2+1}-\frac{2}{x-4}$.

4. $\frac{1}{x^2+1}-\frac{2x-2}{(x^2+1)^2}$.

5. $\frac{1}{3(x-1)}-\frac{x+1}{3(x^2+2)}$.

6. $\frac{1}{2(x-1)}+\frac{x+2}{2(x^2+x+4)}$.

7. $\frac{7}{x-1}+\frac{5x-3}{x^2+x+1}$.

8. $\frac{x}{x^2+2}-\frac{1}{x^2+x+2}$.

Examples CXVI

3. $x^{\frac{1}{2}}+\frac{1}{2}x^{-\frac{1}{2}}y-\frac{1}{8}x^{-\frac{3}{2}}y^2+\frac{1}{16}x^{-\frac{5}{2}}y^3-\frac{5}{128}x^{-\frac{7}{2}}y^4+$ etc.

4. $x^{-2}+2x^{-3}y+3x^{-4}y^2+4x^{-5}y^3+5x^{-6}y^4+$ etc.

5. $x^{-\frac{2}{3}}-\frac{2}{3}x^{-\frac{5}{3}}y+\frac{5}{9}x^{-\frac{8}{3}}y^2-\frac{40}{81}x^{-\frac{11}{3}}y^3+\frac{110}{243}x^{-\frac{14}{3}}y^4-$ etc.

6. $x^{\frac{1}{3}}+\frac{1}{3}x^{-\frac{2}{3}}y-\frac{1}{9}x^{-\frac{5}{3}}y^2+\frac{5}{81}x^{-\frac{8}{3}}y^3-\frac{10}{243}x^{-\frac{11}{3}}y^4+$ etc.

7. $x^{\frac{3}{2}}+\frac{3}{2}x^{\frac{1}{2}}y+\frac{3}{8}x^{-\frac{1}{2}}y^2-\frac{1}{16}x^{-\frac{3}{2}}y^3+\frac{3}{128}x^{-\frac{5}{2}}y^4-$ etc.

Examples CXVII

2. $3x^5-2x^2+(15x^4-4x)h+(30x^3-2)h^2+30x^2h^3+15xh^4+3h^5$.

3. $2x^4-4x^3+x^2-5+(8x^3-12x^2+2x)h$
$+(12x^2-12x+1)h^2+(8x-4)h^3+2h^4$.

Examples CXVIII

2. $16x^4+32x^3y^3+24x^2y^6+8xy^9+y^{12}$.

3. $x^2-2x^4y+4x^6y^2-8x^8y^3+$ etc.

4. $x^{\frac{2}{3}}-\frac{4}{3}x^{-\frac{1}{3}}y^2-\frac{4}{9}x^{-\frac{4}{3}}y^4-\frac{32}{81}x^{-\frac{7}{3}}y^6-$ etc.

Examples CXIX

2. $a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$.

3. $x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7$.

4. $1 + 4x + 6x^2 + 4x^3 + x^4$.

5. $1 - 5y + 10y^2 - 10y^3 + 5y^4 - y^5$.

6. $x^{-2} - 2x^{-3}y + 3x^{-4}y^2 - 4x^{-5}y^3 + 5x^{-6}y^4 -$ etc.

7. $\frac{1}{x^3} + \frac{3y}{x^4} + \frac{6y^2}{x^5} + \frac{10y^3}{x^6} + \frac{15y^4}{x^7} +$ etc.

11. $1 - \frac{1}{2}a^2 - \frac{1}{8}a^4 - \frac{1}{16}a^6 - \frac{5}{128}a^8 -$ etc.

15. $a^6 + 16a^{\frac{29}{6}} + 96a^{\frac{11}{3}} + 256a^{\frac{5}{2}} + 256a^{\frac{4}{3}}$.

16. $x^{\frac{1}{2}} + x^{-\frac{1}{2}} - \frac{x^{-\frac{3}{2}}}{2} + \frac{x^{-\frac{5}{2}}}{2} - \frac{5x^{-\frac{7}{2}}}{8} +$ etc.

17. $1 + 3x^2 + 6x^4 + 10x^6 + 15x^8 +$ etc.

18. $a^{\frac{2}{3}} - \frac{1}{3}a^{-\frac{4}{3}}x^2 - \frac{1}{9}a^{-\frac{10}{3}}x^4 - \frac{5}{81}a^{-\frac{16}{3}}x^6 -$ etc.

19. $\frac{1}{27a^3} + \frac{x^2}{27a^4} + \frac{2x^4}{81a^5} + \frac{10x^6}{729a^6} +$ etc.

22. $2(x^2 + 6x + 1)$.

23. 352.

24. $1 + 4x + 2x^2 - 8x^3 - 5x^4 + 8x^5 + 2x^6 - 4x^7 + x^8$.

Examples CXX

2. $\log y = \frac{2}{3}\log x + \frac{1}{2}[\log(1 + x) + \log(1 - x)]$.

3. $\log y = \frac{1}{2}[\log s + \log(s - a) - \log b - \log c]$.

4. $\log y = \frac{1}{3}[\log a + 2\log b + 4\log c - 5\log d]$.

5. $\log y = 2\log a + \frac{1}{2}[\log(a + b) + \log(a - b) - \log(1 + b)]$.

6. $\log y = \frac{1}{3}[\log x + \log(1 - x)] - \frac{1}{2}\log z$.

7. $\log y = \frac{1}{2}[m\log a + p\log b - t\log c]$.

8. $\log y = \frac{1}{6}[\log(a + b) + \log(a - b) - \log a]$.

9. $\log y = \frac{1}{3}[\log x + \log(x - 4) - \log(x + 1)]$.

11. $dy = \frac{mdx}{x}$.

12. $dy = -\frac{mdx}{1 - x}$.

13. $\frac{3\,mdx}{x}$.

14. $dy = -\frac{mdx}{x}$.

15. $dy = -\frac{2\,mxdx}{a^2 - x^2}$.

16. $dy = \frac{4\,mxdx}{1 + x^2}$.

17. $x = \frac{\log c}{b\log a}$.

18. $x = -\frac{\log b}{\log a}$.

19. $x = \frac{\log a - \log b}{c\log a}$.

20. $x = \log 7$.

21. $x = 2\frac{1}{2}$, $y = \frac{1}{2}$.

22. $x = \frac{1}{2}\left(\frac{\log m}{\log a} + \frac{\log n}{\log b}\right)$.

Examples CXXI

1. 2.902003.
2. 0.935507.
3. 3.891203.
4. 1.915716.
5. $\bar{1}.909930$.
6. 3.930236.
7. 4.896273.
8. 1.929092.
9. $\bar{2}.888089$.
10. 3.938648.
11. 1.895266.
12. $\bar{3}.934229$.

Examples CXXII

1. 8313. **2.** 792.75. **3.** 8.1743. **4.** 86764.6.
5. 79.8252. **6.** .83752. **7.** .0773742. **8.** .0085167.
9. 860900. **10.** 820000. **11.** .008. **12.** 819554.
13. 8636.32. **14.** 85.8052. **15.** 790.3972. **16.** 7.75987.
17. 5.4283. **18.** 5.4641.

Examples CXXIII

5. 1, 1. **6.** 7, 1 ; 2, 4. **7.** 5, 1 ; 19, 6 ; 33, 11 ; etc. **8.** 1, 2 ; 3, 1.
9. Impossible. **10.** 20, 3 ; 39, 7 ; etc. **11.** 17, 1 ; 10, 4 ; 3, 7.
12. 1, 53 ; 3, 40 ; 5, 27 ; 7, 14 ; 9, 1. **13.** 3, 7 ; 8, 21 ; 13, 35 ; etc.
14. Impossible. **15.** 2, 1, 3. **16.** 4, 2, 7.

Examples CXXIV

1. 16, 2 ; 12, 5 ; 8, 8 ; 4, 11. **2.** 11, 15. **3.** 4 ways.
4. 2, 74 ; 15, 55 ; 28, 36 ; 41, 17. **5.** Impossible. 48 ways. **6.** $\frac{3}{2}, \frac{4}{5}, \frac{5}{8}$, etc.
7. 6, 3. **8.** 117, 19 ; 77, 59 ; 37, 99. **9.** 5, 1. **10.** A gives £6 and receives $28. **11.** 1, 6, 8, or 3, 3, 9. **12.** 5, 10, 15.

Examples CXXV

3. $y^6 + 3y^4 + 486 = 0.$ **6.** $y^5 - 2y^4 - 2y^3 + 45y^2 - 12y + 81 = 0.$

Examples CXXVI

2. 19. **3.** 1. **5.** 7. **6.** 12. **7.** 0.

Examples CXXVII

2. 1 pos., 1 neg., 2 imag. **3.** No pos., 1 neg., 4 imag.
4. 1 pos., no neg., 4 imag.
5. Not more than 2 pos., not more than 2 neg., at least 2 imag.
6. Not more than 3 pos., not more than 1 neg., at least 2 imag.
7. 1 pos., not more than 3 neg., at least 2 imag.
8. Not more than 3 pos., 1 neg., at least 2 imag.
9. 1 is the only real root. **10.** 1 and -1 are the only real roots.
11. -1 is the only real root. **12.** No real root.

Examples CXXVIII

2. $2, 3, -4.$ **5.** $7, \pm\sqrt{5}.$ **9.** $1, 1, -1, \frac{1}{2}(-3 \pm \sqrt{5}).$
14. $2, \pm\sqrt{3}, \pm\sqrt{3}.$ **17.** $1, 1, -1, \frac{2}{3}, \frac{3}{2}.$ **20.** $1, 1, 1, 2, 2, -2, -3.$

Examples CXXIX

4. $\pm\sqrt{2}, \pm\sqrt{2}, \pm\sqrt{3}$. **8.** $\pm 2, \pm\sqrt{5}, \pm\sqrt{6}$.

12. $\pm\sqrt{2}, \pm\sqrt{6}, \pm\sqrt{-1}, \pm\sqrt{-2}$. **13.** $0, \pm\sqrt{2}, \pm\sqrt{3}, \pm\sqrt{6}, \pm\sqrt{-2}$.

14. $\pm\sqrt{-1}, \pm\sqrt{-2}, \pm\sqrt{-3}, \pm\sqrt{-5}$. **17.** $3, \pm\sqrt{2}, \pm\sqrt{3}, \pm\sqrt{5}$.

Examples CXXX

7. $4x^5 + 24x^4 + 15x^3 - 80x^2 - 39x + 36 = 0$.

9. $x^6 - 6x^5 + 8x^4 + 18x^3 - 53x^2 + 24x + 20 = 0$.

Examples CXXXI

4. Between 0 and 1, 0 and -1, -5 and -6.

7. Between 3 and 4. Two imag. **14.** Between 0 and 1, 1 and 2, 2 and 3.

18. Between 3.2 and 3.3, 3.4 and 3.5, -1 and -2, -3 and -4.

22. Between 1 and 2, 0 and -1, -1 and -2. Two imag.

Examples CXXXII

2. $2+\sqrt{3}$ and $2-\sqrt{3}$ are each double roots. **4.** $3, 1\pm\sqrt{5}, 1\pm\sqrt{5}$.

6. $\pm\sqrt{2}, \pm\sqrt{2}, 1\pm\sqrt{3}$. **9.** $\pm\sqrt{2}, \pm\sqrt{2}, \pm\sqrt{2}, \pm\sqrt{3}$.

10. $1+\sqrt{3}$ and $(1-\sqrt{3})$ are each quadruple roots.

Examples CXXXIII

3. $x^3 + 20.639x^2 - .343893x + .000043597 = 0$.

4. $x^4 + 11.4x^3 + 36.735x^2 + 36.1965x - .29499375 = 0$.

Examples CXXXIV

3. 2.7824, 4.166, $-.9489$. **4.** 4.117. Two roots imag.

5. $4\frac{1}{8}$, $2\pm\sqrt{3}$; or 4.125, .26795, 3.73205. **6.** -2.2134. Two roots imag.

7. 9.9666. Two roots imag. **8.** 2.25, 3.6055, -3.6055.

9. -2.1768. Two roots imag. **10.** 2.3569, 2.6920, -2.0489.

11. 2.4641, -4.4641. Two roots imag.

12. 4.3166, -2.3166. Two roots imag. **13.** .337, 1.636, -1.691, -4.283.

14. Two roots imag. **15.** 4.5814. Four roots imag.

16. 4.5195. Four roots imag. **17.** 5.874. **18.** .4148. **19.** 106.474.

Examples CXXXVI

4. Between 0 and -1, -2 and -3, two between 2 and 3.

6. Between -1 and -2, 4 and 5, three between 1 and 2.

7. Between -1 and -2, 4 and 5, two between 1 and 2, two imag.

11. Double root between -1 and -2, double root between 1 and 2, two imag.

13. Between -1 and -2, 3 and 4, double root between -1 and -2, double root between 1 and 2.

Examples CXXXVII

2. $1, 1, \frac{1}{2}(1 \pm \sqrt{-3})$. 3. $2 \pm \sqrt{3}, \frac{1}{2}(1 \pm \sqrt{-3})$. 4. $\pm 1, \frac{1}{2}(1 \pm \sqrt{-3})$.

5. $\pm 1, \frac{1}{2}(-7 \pm 3\sqrt{5})$. 6. $\frac{a}{2}(3 \pm \sqrt{5}), \frac{a}{2}(-7 \pm 3\sqrt{5})$.

7. $\pm 1, \frac{1}{a}(1 \pm \sqrt{1 - a^2})$. 8. $1, 3, \frac{1}{3}, -2, -\frac{1}{2}$.

9. $1, \pm\sqrt{-1}, \frac{1}{6}(-1 \pm \sqrt{-35})$. 10. $2, \frac{1}{2}, 2, \frac{1}{2}, \frac{1}{2}(1 \pm \sqrt{-3})$.

Examples CXXXVIII

2. $1, \frac{1}{2}(-1 \pm \sqrt{-3})$. 3. $-1, \frac{1}{2}(1 \pm \sqrt{-3})$. 4. $\pm 1, \pm\sqrt{-1}$.

5. $\pm\sqrt{\pm\sqrt{-1}}$. 6. $-1, \frac{1}{4}\left(1 \pm \sqrt{5} \pm \sqrt{-10 \pm 2\sqrt{5}}\right)$.

7. 3 times the roots of ex. 1. 8. $\pm 1, \pm\sqrt{\frac{1}{2}(-1 \pm \sqrt{-3})}$.

9. $\pm\sqrt{-1}, \pm\sqrt{\frac{1}{2}(1 \pm \sqrt{-3})}$.

Examples CXXXIX

1. Convergent. 3. Convergent. 4. Divergent.

Examples CXL

1. Convergent when $x < 1$, divergent when $x \geqq 1$. 2. Convergent.

3. Convergent when $x \leqq 1$. 4. Convergent. 6. Convergent.

7. Divergent. 8. Convergent for $x > 1$, divergent for $x \leqq 1$.

9. Divergent. 10. Convergent.

11. Convergent for $x \leqq 1$, divergent for $x > 1$. 14. Convergent.

15. Convergent when x is numerically equal to or less than 1, divergent when x is numerically greater than 1.

16. Convergent. 17. Convergent. 18. Divergent.

Examples CXLI

2. $1, 6; 408x^5$. 3. $2, 1; 99x^5$. 4. $-2, 3; 489x^5$.

5. $1, 1, -1; 4x^7$. 6. $-1, -1; 2x^6$. 7. $2, 1, -2; 173x^7$.

Examples CXLII

2. 0. 3. 6, 0. 4. 8, 32.

Examples CXLIII

2. 78. **3.** 225. **4.** $6396\,x''$.

Examples CXLIV

3. $S_n = \frac{1}{3}\left(\frac{11}{6} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right)$; $S_\infty = \frac{11}{18}$.

4. $S_n = \frac{1}{12}\left(\frac{n}{n+1}\right)$; $S_\infty = \frac{1}{12}$. **5.** $S_n = \frac{1}{2}\left(\frac{n}{n+1}\right)$; $S_\infty = \frac{1}{2}$.

7. $S_n = \frac{1}{3} - \frac{1}{2\,n+3}$; $S_\infty = \frac{1}{3}$. **8.** $S_n = 1 - \frac{1}{|\underline{n+1}}$; $S_\infty = 1$.

10. $S_n = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right)$; $S_\infty = \frac{1}{4}$.

11. $S_n = \frac{1}{12} - \frac{1}{4(2\,n+1)(2\,n+3)}$; $S_\infty = \frac{1}{12}$.

13. $S_n = \frac{3}{4} - \frac{2}{n+2} + \frac{1}{2(n+1)(n+2)}$; $S_\infty = \frac{3}{4}$.

14. $S_n = \frac{1}{6} - \frac{1}{n+3} + \frac{2}{(n+3)(n+4)}$; $S_\infty = \frac{1}{6}$.

15. $S_n = \frac{1}{2}\left(1 - \frac{1}{1\cdot 3\cdot 5 \cdots (2\,n+1)}\right)$; $S_\infty = \frac{1}{2}$.

Examples CXLV

2. $\frac{1+11\,x}{1+2\,x-3\,x^2}$. **3.** $\frac{1+x}{1-x-x^2}$. **4.** $\frac{1+3\,x}{(1-x)^2}$. **5.** $\frac{1+x}{1-2\,x-2\,x^2}$.

6. $\frac{2+5\,x}{1+5\,x+4\,x^2}$. **7.** $\frac{1+x}{(1-x)^2}$. **8.** $\frac{3-x-6\,x^2}{1-2\,x-x^2+2\,x^3}$.

9. $\frac{1+2\,x-3\,x^2}{1-x+x^2+x^3}$. **10.** $\frac{1-5\,x+8\,x^3}{1-2\,x-3\,x^2-4\,x^3}$; $p=2,\ q=3,\ r=4$.

Examples CXLVI

2. 400. **3.** 2548. **4.** 1275. **5.** 278,256. **6.** 147.

Examples CXLVII

1. 680. **2.** 1540. **3.** 1505. **4.** 624. **5.** 1240.

6. 2730 **7.** 7490. **8.** 3880. **9.** 36,256.

Examples CXLVIII

2. 10 hr. 19 min. 47.45 sec. **3.** 10 hr. 6 min. 39.1 sec.

4. 3.97905. **5.** $44°\ 58\ 40'$.

Examples CXLIX

1. 336. **2.** 60. **3.** 72, 504, 3024, 15,120, 60,480, 181,440, 362,880, 362,880. **4.** 720. **5.** 210. **6.** 360. **7.** 840. **8.** 40,320. **9.** 360. **10.** 2520, 4,989,600. **11.** 64. **12.** 623,529. **13.** 7,257,600. **14.** (*a*) 5040, (*b*) $\lfloor\underline{7} \times \lfloor\underline{8} \times \lfloor\underline{5} \times \lfloor\underline{3}$. **15.** 1260. **16.** 5040. **17.** (*a*) 576, (*b*) 144. **18.** 576. **19.** 120. **20.** 720. **21.** 120. **22.** 2880. **23.** 181,440. **24.** 408,408. **25.** 4. **26.** 6. **27.** 8. **28.** 15. **29.** 7. **30.** 3. **31.** 13.

Examples CL

1. 15. **2.** 10. **3.** 36, 84, 126, 126, 84, 36, 9, 1. **4.** 15. **5.** 792. **6.** 15,890,700. **7.** 210. **8.** 45. **9.** Loses, \$4.90. **10.** 1716. **11.** 31. **12.** 26. **13.** 4095. **14.** 63; yes. **15.** 3255. **16.** 322. **17.** 35. **18.** 129. **19.** 479. **20.** 2520. **21.** 1980. **22.** (*a*) $\dfrac{\lfloor\underline{52}}{(\lfloor\underline{13})^4}$; (*b*) $\dfrac{\lfloor\underline{52}}{\lfloor\underline{4} \times (\lfloor\underline{13})^4}$. **23.** 945.